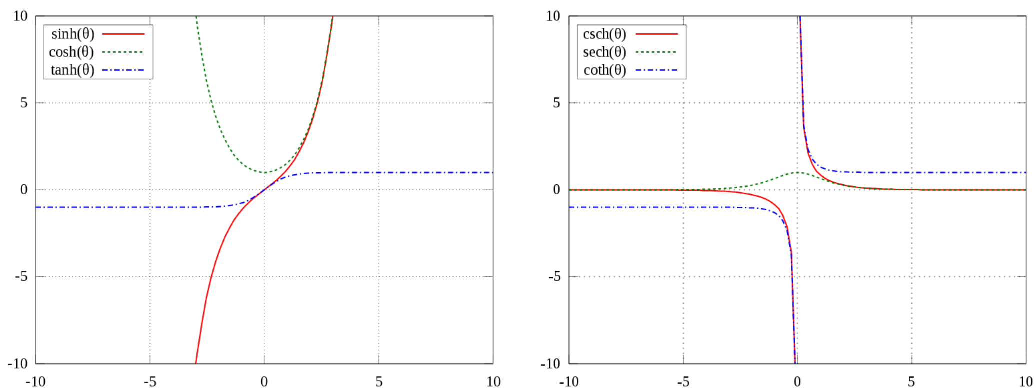
## 7.3 Hyperbolic Functions

Hyperbolic functions are similar to trigonometric functions, and have the following definitions:

- $\sinh x = \frac{1}{2}(e^x - e^{-x})$
- $\cosh x = \frac{1}{2}(e^x + e^{-x})$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\text{csch}x = 1/\sinh x$
- $\text{sech}x = 1/\cosh x$
- $\coth x = 1/\tanh x$

It's also useful to know what they look like




To remember what they look like, just use the definitions and recall what the exponential functions look like and take the average. If you're confused as to what I'm talking about make sure to ask me to explain it.

They are subject to the following identities:

- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \text{sech}^2 x$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

We can prove some of these things, so we may get a better understanding of the identities. Proofs are important, even for engineers!

**Theorem 1.** $\cosh^2 x - \sinh^2 x = 1$.

*Proof.* We go straight to the definition,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\ &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1\end{aligned}$$

□

**Theorem 2.** $1 - \tanh^2 x = \text{sech}^2 x$

*Proof.* Here we simply divide the entire equation by $\cosh^2 x$,

$$\left[\cosh^2 x - \sinh^2 x = 1\right] \frac{1}{\cosh^2 x} \Rightarrow 1 - \tanh^2 x = \text{sech}^2 x.$$

The other identities are proved similar to this one. If you have time, you should try to prove the other identities by yourselves. Even though they wont appear on exams they will help you get a better understanding of the concepts.

□

Here is a nice proof of one of the most important trigonometric identities, and all other identities can be very easily derived through these identities in a similar fashion to the above theorem.

**Theorem 3.** $\sin^2\theta + \cos^2\theta = 1$.

*Proof.* Consider a right triangle and one non-right angle $\theta$. Let the side opposite to $\theta$ be of length $x$, the side adjacent to $\theta$ be of length $y$, and the hypotenuse $z$. Then, $\sin\theta = x/z$ and $\cos\theta = y/z$, and by the Pythagorean theorem $x^2 + y^2 = z^2$, then

$$\sin^2\theta + \cos^2\theta = \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2}{z^2} = \frac{z^2}{z^2} = 1.$$

□

It is important to know the derivatives of hyperbolic functions as well,

- $(\sinh x)' = \cosh x$
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x$
- $(\text{csch} x)' = -\text{csch} x \coth x$
- $(\text{sech} x)' = -\text{sech} x \tanh x$
- $(\coth x)' = -\text{csch}^2 x$

These can all be derived very easily from the definitions.

Ex: $(\cosh\sqrt{x})' = \frac{1}{2\sqrt{x}}(\sinh(\sqrt{x}))$.

## 8.1 Integration Review

Here we do some standard book problems from the section.

4 $I = \int_{\pi/4}^{\pi/3} dx/(\cos^2 x \tan x)$.

**Solution**: $I = \int_{\pi/4}^{\pi/3} dx/\cos x = \int_{\pi/4}^{\pi/3} \sec x dx = \ln|\sec x + \tan x|\Big|_{\pi/4}^{\pi/3} =$ $\ln|2+\sqrt{3}| - \ln|1+\sqrt{2}|$.

18 $I = \int e^{\sqrt{y}} dy/2\sqrt{y}$ **Solution**: Let $u = \sqrt{y}$, then $I = \int 2^u du = 2^u/\ln 2 = 2^{\sqrt{y}}/\ln 2$.

40 $I = \int \sqrt{x} dx/(1+x^3)$. **Solution**: Let $u = x^{3/2} \Rightarrow du = 3\sqrt{x}dx/2$, then $I = \frac{2}{3}\int du/(1+u^2) = \frac{2}{3}\tan^{-1} u = \frac{2}{3}\tan^{-1}(x^{3/2}) + C$.

## 8.2 Integration by parts

The modern notion of integration by parts comes from a beautiful theory of integrals by Riemann and Stieltjes in 1894, soon after which Stieltjes passed away. The idea is we can integrate over certain functions instead of just over $x$. We can think of it as a generalization of "u-sub".

To derive it, consider the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \Rightarrow \mathrm{d}[f(x)g(x)] = f(x)g'(x)\mathrm{d}x + g(x)f'(x)\mathrm{d}x$$
$$\Rightarrow \int \mathrm{d}[f(x)g(x)] = f(x)g(x) = \int f(x)g'(x)\mathrm{d}x + \int g(x)f'(x)\mathrm{d}x$$
$$\Rightarrow \int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\mathrm{d}x.$$

This can be written in the form, which we will use from now on

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u. \tag{1}$$

(1) $I = \int x \sin x \mathrm{d}x$.
**Solution**: Let $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = \sin x \Rightarrow v = -\cos x$. Then,

$$I = -x\cos x + \int \cos x \mathrm{d}x = -x\cos x + \sin x + C.$$

We see here that we generally choose the easiest thing to integrate as $\mathrm{d}v$. We can use ILATE: InverseLogsAlgebraicTrigonometricExponential, to help determine which is easier to integrate. Things get easier to integrate as we go to the right, for example, Exponentials are easier to integrate than Trigonometric functions. But this doesn't always work! So, only use it as a guide, not a rule of thumb.

(2) $I = \int \ln x \mathrm{d}x$.
**Solution**: Let $u = \ln x \Rightarrow \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$. Then,

$$I = x \ln x - \int x \frac{\mathrm{d}x}{x} = x \ln x - x + C.$$

(3) $I = \int t^2 e^t \mathrm{d}t$.
**Solution**: Let $u = t^2 \Rightarrow \mathrm{d}u = 2t\mathrm{d}t$ and $\mathrm{d}v = e^t\mathrm{d}t \Rightarrow v = e^t$. Then,

$$I = t^2 e^t - 2 \int t e^t \mathrm{d}t.$$

Notice, we need to integrate by parts again for the second integral $I_2 = \int t e^t \mathrm{d}t$. Let $u = t \Rightarrow \mathrm{d}u = \mathrm{d}t$ and $\mathrm{d}v = e^t \mathrm{d}t \Rightarrow v = e^t$. Then

$$I_2 = t e^t - \int e^t \mathrm{d}t = t e^t - e^t.$$

Plugging this back into $I$ gives,

$$I = t^2 e^t - 2te^t + 2e^t + C.$$

It may be appealing to do this sort of problem using "tabular integration", however you should avoid using this "method". If you make a mistake using this "method", you will lose a majority of the points. You are better off doing integration by parts twice.

(4) $I = e^x \sin x \mathrm{d}x$.
**Solution**: Let $u = e^x \mathrm{d}x \Rightarrow \mathrm{d}u = e^t \mathrm{d}t$ and $\mathrm{d}v = \sin x \Rightarrow v = -\cos x$. Then,

$$I = -e^x \cos x + \int e^x \cos x \mathrm{d}x.$$

We must do another integration by parts on the second integral. Let $u = e^x \Rightarrow \mathrm{d}u = e^x \mathrm{d}x$ and $\mathrm{d}v = \cos x \Rightarrow v = \sin x$. Then,

$$I_2 = e^x \sin x - \int e^x \sin x \mathrm{d}x.$$

Plugging this into $I$ gives,

$$I = e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x.$$

Now, we add both sides by $\int e^x \sin x \mathrm{d}x$, to get

$$2 \int e^x \sin x \mathrm{d}x = e^x \sin x - e^x \cos x \Rightarrow \int e^x \sin x \mathrm{d}x = \frac{1}{2}(e^x \sin x - e^x \cos x) + C.$$

Notice, for this problem it didn't matter if you chose $e^x$ or $\sin x$ and $\cos x$ as your $u$ or $\mathrm{d}v$. Try this problem the other way around to convince yourself that it works both ways. And as usual, if you're confused about what I'm talking about, please make sure to ask me. It's better to get questions answered early on before you're bombarded with new material.

(5) $I = \int_0^1 \tan^{-1} x\mathrm{d}x$.
**Solution**: Let $u = \tan^{-1} x \Rightarrow \frac{\mathrm{d}x}{1+x^2}$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$. Then,

$$I = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\mathrm{d}x}{1+x^2}.$$

The second integral is our usual u-sub integral where $u = 1 + x^2 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$. Then,

$$I_2 = \frac{1}{2}\int_1^2 \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln u\Big|_1^2 = \ln 2.$$

Plugging this back into $I$ gives,

$$I = x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln u\Big|_1^2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(6) This next example is a test of our abilities to think abstractly. You wont see this sort of thing on the exam, but you'll see things on the exam that use many of the tricks we will use on this example.
Find a reduction formula for $I = \int \sin^n x\mathrm{d}x$.
**Solution**: Let $u = \sin^{n-1} x \Rightarrow \mathrm{d}u = (n-1)\sin^{n-2} x\cos x\mathrm{d}x$ and $\mathrm{d}v = \sin x\mathrm{d}x \Rightarrow v = -\cos x$. Then,

$$\begin{aligned}
\int \sin^n x\mathrm{d}x &= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x\mathrm{d}x \\
&= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\mathrm{d}x \\
&= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x\mathrm{d}x \\
&\Rightarrow n\sin^n x\mathrm{d}x = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\mathrm{d}x \\
&\Rightarrow \sin^n x\mathrm{d}x = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\mathrm{d}x.
\end{aligned}$$

## 8.3 Trigonometric Integration

Lets look at a few examples first and then we'll develop a general strategy.

**Sines and Cosines.**

(1) Consider $\int \cos^3 x \mathrm{d}x$.
**Solution**: We recall the identity: $\cos^2 = 1 - \sin^2 x$. Lets see if we can use this to simplify the problem.

$$\int \cos^3 x \mathrm{d}x = \int \cos x[1 - \sin^2 x]\mathrm{d}x = \int \cos x \mathrm{d}x - \int \sin^2 x \cos x \mathrm{d}x.$$

Now, the first integral is easy and the second integral we solve via u-sub where $u = \sin x \Rightarrow \mathrm{d}u = \cos x \mathrm{d}x$.

$$\int \cos^3 x \mathrm{d}x = \sin x - \int u^2 \mathrm{d}u = \sin x - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

(2) $\int \sin^5 x \cos^2 x \mathrm{d}x$.
**Solution**: Lets use the same strategy as above, except this time on $\sin x$.

$$\int \sin^5 x \cos^2 x \mathrm{d}x = \int (\sin^2 x)^2 \sin x \cos^2 x \mathrm{d}x = \int (1 - \cos^2 x)^2 \cos^2 x \sin x \mathrm{d}x.$$

We can go straight to u-sub with $u = \cos x \Rightarrow \mathrm{d}u = -\sin x$,

$$\int \sin^5 x \cos^2 x \mathrm{d}x = -\int (1-u^2)^2 u^2 \mathrm{d}u = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

(3) $\int_0^\pi \sin^2 x \mathrm{d}x$
**Solution**: For this problem if we used the identity we used for the past two problems we would be going in circles, so we use another identity - the double angle formula: $\cos 2x = 1 - 2\sin^2 x$,

$$\int_0^\pi \sin^2 x \mathrm{d}x = \frac{1}{2}\int_0^\pi (1 - \cos 2x)\mathrm{d}x = \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^\pi = \frac{\pi}{2}$$

(4) $\int \sin^4 x \mathrm{d}x$
**Solution**: This is similar to the above problem,

$$\int \sin^4 x \mathrm{d}x = \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \mathrm{d}x = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\mathrm{d}x$$
$$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] \mathrm{d}x = \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

*Strategies for* $\int \sin^m x \cos^n x dx$.

(1) If the power of the cosine term is odd (i.e. $n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$,

$$\begin{aligned} \int \sin^m x \cos^{2k+1} \mathrm{d}x &= \int \sin^m x (\cos^2 x)^k \cos x \mathrm{d}x \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \mathrm{d}x. \end{aligned} \tag{2}$$

Then substitute $u = \sin x \Rightarrow \mathrm{d}u = \cos x$.

(2) If the power of the sine term is odd (i.e. $m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$,

(3)
$$\int \sin^{2k+1} \cos^n x \mathrm{d}x = \int (\sin^2 x)^k \cos^n x \mathrm{d}x = \int (1 - \cos^2 x)^k \cos^n x \sin x \mathrm{d}x.$$

Then substitute $u = \cos x \Rightarrow \mathrm{d}u = -\sin x$.

(3) If the powers of both sine and cosine are even, use the double-angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin x \cos x = \frac{1}{2} \sin 2x.$$

**Tangents and Secants.**

(1) $\int \tan^6 x \sec^4 x \mathrm{d}x$.
**Solution**: We recall the identity $\sec^2 x = 1 + \tan^2 x$, and see where this takes us

$$\int \tan^6 x \sec^4 x \mathrm{d}x = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \mathrm{d}x.$$

Then we substitute $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x \mathrm{d}x$, then

$$\int \tan^6 x \sec^4 x \mathrm{d}x = \int u^6 (1+u^2) \mathrm{d}u = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

(2) $\int \tan^5\theta \sec^7\theta \mathrm{d}\theta$.

**Solution**: Here lets try using the other identity: $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^5\theta \sec^7\theta \mathrm{d}\theta = \int \tan^4\theta \sec^6\theta \sec\theta \tan\theta \mathrm{d}\theta = \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \mathrm{d}\theta.$$

We employ the u-sub $u = \sec\theta \Rightarrow \mathrm{d}u = \sec\theta\tan\theta\mathrm{d}\theta$,

$$\int \tan^5\theta \sec^7\theta \mathrm{d}\theta = \int (u^2-1)^2 u^6 \mathrm{d}u = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C = \frac{1}{11}\sec^{11}x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C$$

*Strategies for* $\int \tan^m x \sec^n x dx$.

(1) If the power of the secant term is even (i.e. $n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned}\int \tan^m x \sec^{2k} x \mathrm{d}x &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \mathrm{d}x \\ &= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \mathrm{d}x. \end{aligned} \tag{4}$$

Then substitute $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x \mathrm{d}x$.

(2) If the power of the tangent term is odd (i.e. $m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$,

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \mathrm{d}x &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x. \end{aligned} \tag{5}$$

Then substitute $u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \mathrm{d}x$

*Useful Integrals.*

These integrals are also pretty easy to derive if you forget them,

$$\int \tan x \mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C. \tag{6}$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C. \tag{7}$$

(1) $\int \tan^3 x \mathrm{d}x$.

**Solution**: We use the identity $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^3 x \mathrm{d}x = \int \tan x(\sec^2 x - 1)\mathrm{d}x = \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

(2) $\int \sec^3 x \mathrm{d}x$.

**Solution**: We integrate by parts with $u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \mathrm{d}x$ and $\mathrm{d}v = \sec^2 x \Rightarrow v = \tan x$, then

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \sec x \tan x - \int \sec x \tan^2 x \mathrm{d}x = \sec x \tan x - \int \sec x (\sec^2 x - 1) \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \int \sec x \mathrm{d}x = \sec x \tan x - \int \sec^3 x \mathrm{d}x + \ln|\sec x + \tan x| \\ &\Rightarrow \int \sec^3 x \mathrm{d}x = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C.\end{aligned}$$