Cross-sectional area $A(x)$: $$V = \int_a^b A(x)\mathrm{d}x \quad (1)$$

Disk about x-axis: $$V = \int_a^b \pi R(x)^2 \mathrm{d}x \quad (2)$$

Disk about y-axis: $$V = \int_a^b \pi R(y)^2 \mathrm{d}y \quad (3)$$

Washers about x-axis: $$V = \int_a^b \pi[R(x)^2 - r(x)^2]\mathrm{d}x \quad (4)$$

Washers about y-axis: $$V = \int_a^b \pi[R(y)^2 - r(y)^2]\mathrm{d}y \quad (5)$$

Cylindrical Shells about y-axis: $$V = \int_a^b 2\pi x f(x)\mathrm{d}x \quad (6)$$

Cylindrical Shells about x-axis: $$V = \int_a^b 2\pi y f(y)\mathrm{d}y \quad (7)$$

Integration by parts: $$\int u\mathrm{d}v = uv - \int v\mathrm{d}u. \quad (8)$$

Strategies for $\int \sin^m x \cos^n x\, dx$.

(1) If the power of the cosine term is odd (i.e. $n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$,

$$\begin{aligned} \int \sin^m x \cos^{2k+1} \mathrm{d}x &= \int \sin^m x (\cos^2 x)^k \cos x \mathrm{d}x \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \mathrm{d}x. \end{aligned} \tag{9}$$

Then substitute $u = \sin x \Rightarrow \mathrm{d}u = \cos x$.

(2) If the power of the sine term is odd (i.e. $m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$,

$$\int \sin^{2k+1} \cos^n x \mathrm{d}x = \int (\sin^2 x)^k \cos^n x \mathrm{d}x = \int (1 - \cos^2 x)^k \cos^n x \sin x \mathrm{d}x. \tag{10}$$

Then substitute $u = \cos x \Rightarrow \mathrm{d}u = -\sin x$.

(3) If the powers of both sine and cosine are even, use the double-angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin x \cos x = \frac{1}{2} \sin 2x.$$

Strategies for $\int \tan^m x \sec^n x\, dx$.

(1) If the power of the secant term is even (i.e. $n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned} \int \tan^m x \sec^{2k} x \mathrm{d}x &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \mathrm{d}x \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \mathrm{d}x. \end{aligned} \tag{11}$$

Then substitute $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x \mathrm{d}x$.

(2) If the power of the tangent term is odd (i.e. $m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$,

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x \mathrm{d}x &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x. \end{aligned} \tag{12}$$

Then substitute $u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \mathrm{d}x$

Useful Integrals.

These integrals are also pretty easy to derive if you forget them,

$$\int \tan x \mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C. \tag{13}$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C. \tag{14}$$

Strategies for trig-sub.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\pi/2 \leq \theta \leq \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \leq \theta < \pi/2,\ \pi \leq \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

Midpoint rule:

$$\int_a^b f(x)\mathrm{d}x \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]; \tag{15}$$

$$x_i^* = \frac{1}{2}(x_i + x_{i+1}),\ \Delta x = \frac{b-a}{n}$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_M| \leq \frac{K(b-a)^3}{24n^2};\ |f''(\xi)| \leq K,\ \xi \in [a,b]. \tag{16}$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a,b]$.

Trapezoid rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]; \tag{17}$$

$$\Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x.$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_T| \leq \frac{K(b-a)^3}{12n^2};\ |f''(\xi)| \leq K,\ \xi \in [a,b]. \tag{18}$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a,b]$.

Simpson's rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)]; \tag{19}$$

$$\Delta x = \frac{b-a}{n},\ n \geq 4 \quad \text{and } n \text{ must be even.}$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_S| \leq \frac{K(b-a)^5}{180n^4};\ |f^{(4)}(\xi)| \leq K,\ \xi \in [a,b]. \tag{20}$$

Where $|f^{(4)}(\xi)|$ is just the maximum of the fourth derivative in $[a, b]$.

Partial fractions.

Case 1.

Suppose Q is a product of distinct linear factors, i.e. $Q = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. Then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}. \tag{21}$$

Case 2.

Suppose Q is a product of linear factors, some of which are repeated. Then, the repeated factors are of this form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}. \tag{22}$$

Case 3.

Suppose Q is a product of quadratic factors with no repeats, i.e. $Q = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)$. Then,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)} \\ &= \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}.\end{aligned} \tag{23}$$

Case 4.

Suppose Q is product of factors that include repeated quadratic factors. Then the repeated quadratic factors will be of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax^2 + bx + c)^r} &= \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} \\ &\quad + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}.\end{aligned} \tag{24}$$

Improper Integrals.

Case 1: Infinite Intervals

a) If $\int_a^t f(x)\mathrm{d}x$ exists for all $t \geq a$, then $\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\mathrm{d}x$.

b) If $\int_t^b f(x)\mathrm{d}x$ exists for all $t \leq b$, then $\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\mathrm{d}x$.

Definition 1. If $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are convergent if the limit exists, and divergent if the limit does not exist.

$$c)\text{If } \int_a^\infty f(x)\mathrm{d}x and \int_{-\infty}^a f(x)\mathrm{d}x \text{ are convergent },$$
$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

Case 2: Integrands with Discontinuities.

a) If f is continuous in $[a, b)$ and discontinuous at $x = b$,

then $\int_a^b f(x)\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\mathrm{d}x$.

b) If f is continuous in $(a, b]$ and discontinuous at $x = a$,

then $\int_a^b f(x)\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\mathrm{d}x$.

Definition 2. The integral $\int_a^b f(x)\mathrm{d}x$ is said to be convergent if the limit exists, and divergent if the limit does not exist.

c) If f has a discontinuity at $c \in [a, b]$ and $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ both converge, then

$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$.

$$\text{Taylor series:} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{25}$$

$$\text{Remainder:} \qquad |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}. \tag{26}$$

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \tag{27}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \tag{28}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{29}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots . \tag{30}$$

Parametric derivative: $$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)}{f'(t)};\ \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0. \tag{31}$$

Second derivative: $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)/f'(t)}{f'(t)} = \frac{g'(t)}{f'(t)^2};\ \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0. \tag{32}$$

Arc Length: $$L = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t = \int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\mathrm{d}t. \tag{33}$$

Surface Area about x-axis: $$\mathrm{SA} = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t \tag{34}$$

Surface Area about y-axis: $$\mathrm{SA} = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t \tag{35}$$

Polar Coordinates: $$x = r\cos\theta,\ y = r\sin\theta;\ r^2 = x^2 + y^2,\ \theta = \tan^{-1}(y/x) \tag{36}$$

Area of a wedge: $$A = \int_\alpha^\beta \frac{1}{2} r^2 \mathrm{d}\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 \mathrm{d}\theta. \tag{37}$$

Polar Arc Length: $$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta. \tag{38}$$