FALL 2015 SOLUTIONS

(1) (a) We take the limit,

$$\lim_{n\to\infty} \frac{1+e^{-n}}{1/e+e^{-2n}} = e,$$

and hence it converges.

(b) We take the limit,

$$\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\sin(1/n)}{-1/n^2} = \lim_{n\to\infty} \sin(1/n) = 0,$$

and hence it converges.

(2) (a) $I \approx -\frac{1}{8}\left[0 + 2\cdot\frac{\sqrt{2}}{2} + 2 + 2\cdot\frac{\sqrt{2}}{2} + 0\right] = -\frac{\sqrt{2}+1}{4}$.

(b) $I \approx -\frac{1}{12}\left[0 + 4\cdot\frac{\sqrt{2}}{2} + 2 + 4\cdot\frac{\sqrt{2}}{2} + 0\right] = -\frac{2\sqrt{2}+1}{6}$.

(3) (a) We use partial fractions,

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} = \frac{x^2+3}{(x+1)^3} \Rightarrow A(x+1)^2 + B(x+1) + C = Ax^2 + 2Ax + Bx + A + B + C$$
$$\Rightarrow Ax^2 + (2A+B)x + (A+B+C) = x^2+3 \Rightarrow A = 1, B = -2, C = 4.$$

So the integral is,

$$\int \left(\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{4}{(x+1)^3}\right) = \ln|x+1| + \frac{2}{x+1} - \frac{2}{(x+1)^2}.$$

(b) Here we use u-sub, with $u = 4x^2 - 1 \Rightarrow du = 8xdx$,

$$I = \frac{1}{8}\int \frac{du}{u^{3/2}} = -\frac{1}{4}u^{-1/2} + C = -\frac{1}{4}(4x^2-1)^{-1/2} + C.$$

(4) We use partial fractions,

$$\frac{A+Bx}{x^2+1} + \frac{C+Dx}{(x^2+1)^2} = \frac{x^2+2x+1}{(x^2+1)^2}$$
$$\Rightarrow (A+Bx)(x^2+1) + C + Dx = Bx^3 + Ax^2 + (B+D)x + (A+C) = x^2+2x+1$$
$$\Rightarrow B = 0, A = 1, D = 2, C = 0.$$

Hence, the integral is

$$I = \int \left(\frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2}\right) dx = \tan^{-1} x - \frac{1}{x^2+1} + C.$$

(5) Here we use u-sub, with $u = x^2 + 4 \Rightarrow du = 2xdx \Rightarrow x^2 = u - 4$,

$$I = \frac{1}{2}\int \frac{u^2 - 8u + 16}{u^{5/2}} du = \frac{1}{2}\int \left(u^{-1/2} - 8u^{-3/2} + 16u^{-5/2}\right)$$
$$= u^{1/2} + 8u^{-1/2} - \frac{16}{3}u^{-3/2} + C = (x^2+4)^{1/2} + 8(x^2+4)^{-1/2} - \frac{16}{3}(x^2+4)^{-3/2} + C.$$

(6) We use trig sub, $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$,

$$I = \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C.$$

(7) First we use integration by parts, $u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2}$, $dv = x^3 dx \Rightarrow x^3/3$.

$$I = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx.$$

We need to use long division on the second integral,

$$\int \frac{x^3}{1+x^2} = \int x - \frac{x}{x^2+1} = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1|.$$

Then the full integral is,

$$I = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln|x^2+1| + C.$$

(8) (a) Notice the singularity is at $\pi/2$, then we have to do one integral at a time,

$$\lim_{t\to\pi/2}\int_0^t \frac{\cos x}{1-\sin x}dx = \lim_{t\to\pi/2} |1-\sin x|\Big|_0^t = \infty$$

Since one integral diverges, the entire integral diverges.

(b) Here we use u-sub, $u = \ln x \Rightarrow du = dx/x$. Then our integral is,

$$I = \lim_{t\to\infty}\int_1^t \frac{du}{u^2} = \lim_{t\to\infty} -\frac{1}{u}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1$$

Hence, the integral converges.

(9) (a) We use direct comparison since $e^{1/x}/(x^2+4) \leq e/x^2$ on the interval $[1,\infty)$. Now, $\int_1^\infty e/x^2 dx$ converges since $p > 1$, hence the original integral converges by DCT.

(b) Here we use limit comparison. By looking at the highest power terms we assume that the comparison will be to $1/x$, to prove this we take the limit of the ratios,

$$\lim_{x\to\infty}\frac{x^2/\sqrt{x^6+6}}{1/x} = \lim_{x\to\infty}\frac{x^3}{\sqrt{x^6+6}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+6/x^2}} = 1.$$

And, $\int_1^\infty dx/x$ diverges since $p \leq 1$, hence the original integral diverges by LCT.