

## SPRING 2011 SOLUTIONS

- (1) Since the region is revolved around the  $x$ -axis and it asks us to use cylindrical shells, we integrate over  $y$ . Recall, cylindrical shells have a surface area of  $A = 2\pi rh$ , and  $r = y$  and  $h = y^2$ . So, we have,

$$V = 2\pi \int_0^{\sqrt{6}} y^3 dy = 2\pi \left[ \frac{1}{4}y^4 \right]_0^{\sqrt{6}} = 18\pi.$$

- (2) Differentiating gives,  $\frac{dx}{dy} = \frac{1}{3} \left( \frac{3}{2}y^{1/2} - \frac{3}{2}y^{-1/2} \right)$ . Squaring gives,  $\left( \frac{dx}{dy} \right)^2 = \frac{1}{4} \left( y - 2 + \frac{1}{y} \right)$ . Notice, that when we add 1 we get a “whole square”,

$$\left( \frac{dx}{dy} \right)^2 + 1 = \frac{1}{4} \left( y + 2 + \frac{1}{y} \right) = \frac{1}{4} \left( y^{1/2} + y^{-1/2} \right)^2.$$

Now we recall that the arc length formula is  $L = \int_a^b \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy$ . But notice that the quantity inside the radical:  $1 + \frac{dx}{dy}$ , is just Eq 1. So, we take the square root of Eq 1 and integrate that,

$$L = \frac{1}{2} \int_1^9 \left( y^{1/2} + y^{-1/2} \right) dy = \frac{1}{3} \left( y^{3/2} + 3y^{1/2} \right) \Big|_1^9 = \frac{32}{3}.$$

- (3) Recall the area for a surface of revolution is,  $SA = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx$ . Taking the derivative gives  $f'(x) = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x) = \frac{1-x}{\sqrt{2x-x^2}}$ . Plugging this into the formula gives,

$$\begin{aligned} SA &= 2\pi \int_0^2 \sqrt{2x-x^2} \sqrt{1 + \frac{(1-x)^2}{2x-x^2}} dx \\ &= 2\pi \int_0^2 \sqrt{2x-x^2+1-2x+x^2} dx = 2\pi \int_0^2 \sqrt{1} dx = 4\pi. \end{aligned}$$

- (4) We should first draw the region to see what it looks like. For this problem it's easiest to use the washer method integrating in  $x$ . We notice that our big radius will be  $R(x) = 2 - x^2$  and the smaller radius will be  $r(x) = x^2$ . Then, we recall the formula for the washer method is  $V = \pi \int_a^b [R(x)^2 - r(x)^2] dx$ . Plugging our quantities in gives,

$$V = \pi \int_{-1}^1 [(2-x^2)^2 - x^4] dx = 2\pi \int_0^1 (4-4x^2) dx = 2\pi \left[ 4x - \frac{4}{3}x^3 \right]_0^1 = \frac{16}{3}\pi.$$

- (5) This is similar to the example problem we did when we covered work. Consider the coordinate system to be  $x = 0$  at the top and  $x = 100$  at the end of the fully stretched cable. Notice that the weight of a small piece of cable will be  $F_i = 5\Delta x_i$ , then the work it takes to move this piece up to the top will be  $W_i = 5x_i\Delta x_i$ , where  $x_i$  is the location of that piece (i.e. distance from the top). Then, we can just integrate this,

$$W = \int_0^{100} 5x dx = \frac{5}{2}x^2 \Big|_0^{100} = \frac{5}{2} \times 10^4.$$

- (6) This is similar to a problem in Quiz 1. We know the area of a semicircle is  $A = \frac{1}{2}\pi r^2$ . The radii here are just  $r = y = \sqrt{\frac{x}{1+x^2}}$ , so  $A(x) = \frac{\pi}{2} \frac{x}{1+x^2}$ . Then, integrating gives

$$V = \frac{\pi}{2} \int_1^2 \frac{x dx}{1+x^2} = \frac{\pi}{4} \int_2^5 \frac{du}{u} = \frac{\pi}{4} \ln u \Big|_2^5.$$

- (7) (a)

$$\frac{d}{dx} [\sqrt{x} \cosh(\sqrt{x})] = \frac{1}{2\sqrt{x}} \cosh(\sqrt{x}) + \frac{1}{2} \sinh(\sqrt{x}).$$

- (b) We convert  $\tanh(3x) = \sinh(3x)/\cosh(3x)$ , and use  $u = \cosh(3x) \Rightarrow du = 3 \sinh(3x)$ ,

$$\int \tanh(3x) dx = \int \frac{\sinh(3x)}{\cosh(3x)} dx = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| = \frac{1}{3} \ln |\cosh(3x)|.$$

- (8) For any work problem like this, we imagine breaking our region up into small pieces and finding the work it takes to move those small pieces. We go back to the fundamental equation for work:  $W_i = F_i d_i$ . With the coordinate system given, the distance a piece will move will be  $d_i = 5 - y_i$  because the height of the tank is  $y = 4$ ft and we want to pump the water to a height 1ft above, and since we are pumping the fluid out of the top we will keep everything in terms of  $y$ .

Now, we know  $F_i = \rho V_i = 40V_i$  because we are using English units, so  $W_i = 40V_i(5 - y_i)$ . Now, we must find the volume of each little piece. Notice for this problem it is highly preferable to think of the pieces as circular cylinders, so the volume will be  $V_i = \pi r_i^2 h_i$ . We think of these cylinders as having a small height, say  $h_i = \Delta y_i$ . Now, if we drew the region properly, we easily see that the radius is  $r_i^2 = x^2 = \frac{y_i}{4}$ , then the volume is  $V_i = \pi \frac{y_i}{4} \Delta y_i$ . Plugging this into the formula for work gives,  $W_i = 40\pi \frac{y_i}{4} (5 - y_i) \Delta y_i$ . Then we integrate,

$$W = 10\pi \int_0^4 (5y - y^2) dy = 10\pi \left[ \frac{5}{2}y^2 - \frac{1}{3}y^3 \right]_0^4 = 10\pi \left( 40 - \frac{64}{3} \right) = \frac{560}{3}\pi.$$

SPRING 2011 EXAM 2 SOLUTIONS FOR "BY PARTS" AND "TRIG INTEGRATION"

- (1a) This is a typical integration by parts problem. Use  $u = x \Rightarrow du = dx$  and  $dv = \cos x \Rightarrow v = \sin x$ .  
 (2a) There are two equivalent ways of solving this. I will solve it one way, and you should solve it the other way and then ask me in class about it and answer the question that I will pose.

First we use the trig identity  $\tan^2 x = 1 + \sec^2 x$ ,  $\int \tan^3 x dx = -\int \tan x dx + \int \tan x \sec^2 x dx$ . Then we use  $u = \tan x \Rightarrow du = \sec^2 x dx$  for the second integral, then

$$I = -\int \tan x dx + \int \tan x + \sec^2 x dx = \ln |\cos x| + \int u du = \ln |\cos x| + \frac{1}{2} \tan^2 x + C.$$

- (4) (a) This is a typical trig-sub problem, with the substitution  $x = \sin \theta$ ,

$$I = \int \frac{5 \cos \theta d\theta}{5^2 \sin^2 \theta \sqrt{25 - 25 \sin^2 \theta}} = \int \frac{5 \cos \theta d\theta}{5^2 \sin^2 \theta (5 \cos \theta)} = \frac{1}{25} \int \csc^2 \theta = -\frac{1}{25} \cot \theta + C.$$

Now, we must solve for  $\cot \theta$ . Notice  $\sin \theta = x/5$ , so the adjacent side is  $\sqrt{25 - x^2}$ , then  $\cot \theta = \frac{\sqrt{25 - x^2}}{x}$ , so

$$I = -\frac{\sqrt{25 - x^2}}{25x} + C.$$

- (b) This is a typical integration by parts problem where  $u = \ln x \Rightarrow du = dx/x$ , and  $dv = dx/x^3 = x^{-3} dx \Rightarrow v = -x^{-2}/2$ , then

$$\int \frac{\ln x}{x^3} dx = \frac{-1}{2} x^{-2} \ln x + \frac{1}{2} \int x^{-3} dx = \frac{-1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} + C.$$

- (7) This is a typical trig-sub problem, where we use  $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$ .

$$\int \frac{dx}{(1 + x^2)^{5/2}} = \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^{5/2}} = \int \frac{\sec^2 \theta}{\sec^5 \theta} d\theta = \int \frac{d\theta}{\sec^3 \theta} = \int \cos^3 \theta d\theta.$$

This is our usual trig integral where we use  $\cos^2 \theta = 1 - \sin^2 \theta$ , then

$$\int \cos^3 \theta d\theta = \int (1 - \sin^2 \theta) \cos \theta d\theta = \int \cos \theta d\theta - \int \sin^2 \theta \cos \theta d\theta = \sin \theta - \int \sin^2 \theta \cos \theta d\theta.$$

The first integral is easy, and for the second integral we use u-sub with  $u = \sin \theta \Rightarrow du = \cos \theta d\theta$ , then

$$\int \sin^2 \theta \cos \theta d\theta = \int u^2 du = \frac{1}{3} u^3 = \frac{1}{3} \sin^3 \theta.$$

Then, we get

$$I = \sin \theta - \frac{1}{3} \sin^3 \theta + C.$$

Now, we must substitute back. Since  $\tan \theta = x$ , the hypotenuse will be  $\sqrt{x^2 + 1}$ , then  $\sin \theta = x/\sqrt{x^2 + 1}$ , and plugging this back in gives

$$I = \frac{x}{\sqrt{x^2 + 1}} - \frac{1}{3} \left( \frac{x}{\sqrt{x^2 + 1}} \right)^3 \theta + C.$$

#### FALL 2011 SOLUTIONS

- (1) Differentiating gives  $\frac{dy}{dx} = \frac{1}{2}x^{1/2}$ . Notice this is not a problem that we can turn into a “whole square”, so we go ahead and plug this into our arc length formula,

$$L = \int_0^{12} \sqrt{\frac{x}{4} + 1} dx = 4 \int_1^4 u^{1/2} du = \frac{8}{3} u^{3/2} \Big|_1^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}.$$

- (2) Since we are revolving around the  $y$ -axis, the standard formulas will be in terms of  $y$ . We differentiate to get  $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$ . Plugging this into the formula gives,

$$SA = 2\pi \int_2^6 \sqrt{y} \sqrt{\frac{1}{4y} + 1} dy = 2\pi \int_2^6 \sqrt{\frac{1}{4} + y} dy = 2\pi \int_{9/4}^{25/4} u^{1/2} = 2\pi \frac{2}{3} u^{3/2} \Big|_{9/4}^{25/4} = \frac{49\pi}{3}.$$

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- (3) Since we are using cylindrical shells and revolving around the  $y$ -axis, our integration will be in terms of  $x$ . Our radius is just  $r = x$  and our height is just  $h = y = \sin(\pi x^2)$ . Plugging into the formula gives,

$$V = 2\pi \int_0^1 x \sin(\pi x^2) dx = \int_0^\pi \sin(u) du = -\cos(u) \Big|_0^\pi = 2.$$

- (4) This is fairly easy to sketch, and when we do sketch it, we see that each side of the square will be of length  $L = 2y = 2(1 - x^2)$ , so the area will be  $A = (2y)^2 = 4(1 - x^2)^2 = 4(1 - 2x^2 + x^4)$ . Integrating this gives,

$$V = 4 \int_0^1 (1 - 2x^2 + x^4) dx = 4 \left[ x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_0^1 = \frac{32}{15}.$$

- (5) We did this in class, so I'll just give you guys the integrals we got using disks and washers respectively.  
(a)

$$V = \pi \int_0^1 (e^x - 1)^2 dx = \pi \int_0^1 (e^{2x} - 2e^x + 1) dx = \pi \left[ \frac{1}{2}e^{2x} - 2e^x + x \right]_0^1$$

(b)

$$V = \pi \int_0^1 (e^{2x} - 1) dx = \pi \left[ \frac{1}{2}e^{2x} - x \right]_0^1.$$

- (6) This is similar to the cable problems we did. First we calculate the work to pull the bucket up since that is easy because the weight of the bucket + water is constant, unlike the rope. Since the weight is constant, the work is  $W_b = (30)(12) = 360\text{ft}\cdot\text{lb}$ .

Now we calculate the work done to pull up the rope. Since the density of the rope is  $\rho = .5\text{lb}/\text{ft}$ , the work done to pull it up will be,

$$W_r = \int_0^{12} \frac{x}{2} dx = \frac{x^2}{4} \Big|_0^{12} = 36.$$

Then,  $W = 360 + 36 = 396$ .

- (7) This is similar to the other tank problem. The volume will be,  $V = \pi r^2 h = \pi x^2 h = \pi \sqrt{y} \Delta y$ , then the force is  $F = 25\pi \sqrt{y} \Delta y$ . Then we get the integral for the work,

$$W = 25\pi \int_0^1 \sqrt{y}(3 - y) dy = 25\pi \left[ 2y^{3/2} - \frac{2}{5}y^{5/2} \right]_0^1 = 40\pi.$$

(3a) This is a typical trig integral problem where  $x = \frac{1}{3} \tan \theta \Rightarrow dx = \frac{1}{3} \sec^2 \theta d\theta$ .

$$\int \frac{dx}{\sqrt{1+9x^2}} = \frac{1}{3} \int \frac{\sec^2 \theta d\theta}{\sec \theta} = \frac{1}{3} \ln |\sec \theta + \tan \theta| + C.$$

Notice  $\tan \theta = 3x$ , then the hypotenuse is  $\sqrt{9x^2 + 1}$ , so  $\sec \theta = \sqrt{9x^2 + 1}$ , then

$$I = \frac{1}{3} \ln \left| \sqrt{9x^2 + 1} + 3x \right| + C.$$

(4) (a) We first integrate by parts using  $u = \tan^{-1} x \Rightarrow du = 1/(1+x^2)$ , and  $dv = x dx \Rightarrow v = x^2/2$ ,

$$\int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx.$$

Notice the last integral is a trig-sub where we use  $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$ ,

$$\int \frac{x^2}{1+x^2} dx = \int \frac{\tan^2 \theta \sec^2 \theta}{\sec^2 \theta} d\theta = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta = x - \tan^{-1} x$$

Then plugging back in gives,

$$I = \frac{1}{2} x^2 \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C.$$

(b) This is a typical trig-sub problem where  $x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta$ .

$$\begin{aligned} \int \frac{x^2 dx}{(4-x^2)^{3/2}} &= \int \frac{4 \sin^2 \theta (2 \cos \theta) d\theta}{(4-4 \sin^2 \theta)^{3/2}} = \int \frac{4 \sin^2 \theta (2 \cos \theta) d\theta}{8 \cos^3 \theta} = \frac{\sin^2 \theta d\theta}{\cos^2 \theta} \\ &= \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C. \end{aligned}$$

Now we must substitute back. Since  $\sin \theta = x/2$ , then the adjacent side is  $\sqrt{4-x^2}$ , so  $\tan \theta = x/\sqrt{4-x^2}$ ,

$$I = \frac{x}{\sqrt{4-x^2}} - \sin^{-1} \frac{x}{2} + C.$$

(5b) We solve this problem by parts using  $u = x \Rightarrow du = dx$ , and  $dv = \cosh x dx \Rightarrow v = \sinh x$ ,

$$\int x \cosh x dx = x \sinh x - \int \sinh x dx = x \sinh x - \cosh x + C.$$