10.9 Convergence of Taylor Series

Since we have so many examples for these sections and it's usually a simple matter of recalling the formula and plugging in for it, I'll simply provide the answers. Also, many of these problems are in the book so I would suggest looking up the problems in the book as well.

36) $x\sin^2 x = x^3 - x^5/3 + 2x^7/45 + \cdots$ converges everywhere by the theorem on products of power series in in a previous section.

Theorem 1. *Taylor: If f has derivatives up to order n at a neighborhood of $x = a$, then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{a!}(x-a)^n + R_n(x). \quad (1)$$

Where,

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad \textit{for some } \xi \textit{ between } x \textit{ and } a. \quad (2)$$

Theorem 2. *Remainder: If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}. \quad (3)$$

(1) We may use remainders to find an alternate proof of the convergence of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Note, this is not a very efficient proof, so the proof by ratio test is preferred. But nevertheless, we find the remainder and take it's limit,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{e^{\xi}}{(n+1)!}x^{n+1} = 0 \qquad \text{for any } x.$$

Since this is true for all x, it must converge everywhere.

(2) Find the Taylor series of e^x about $x = 2$, and bound it's remainder.
Solution: For this problem lets find the bound in the general case, however on the exam you will be asked to find it for a certain domain.

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n; \ |R_n(x)| = \left|\frac{e^{\xi}}{(n+1)!}(x-2)^{n+1}\right| \le \frac{(x-2)^{n+1}}{(n+1)!}\begin{cases} e^2 & x < 2, \\ e^x & x > 2; \end{cases}$$

(3) Find the bound for the remainder of $\sin x$.
Solution: We found the taylor series of $\sin x$ in a previous section. The bound on the remainder will be,

$$|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}.$$

We choose $M = 1$ because $\sin x \le 1$ for all x.

(4) Find the Taylor series of $x\cos x$.
Solution: We found the Taylor series of $\cos x$ already, which is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$. Therefore, $x\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$.

It is useful to know a few common Taylor series. Of course, you should be able to derive these.

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \tag{4}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \tag{5}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{6}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots . \tag{7}$$

(1) Evaluate $\int e^{-x^2} \mathrm{d}x$ as a series and calculate $\int_0^1 e^{-x^2} \mathrm{d}x$ correct to within .001.
Solution: We know $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, then

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots .$$

We can integrate this term by term,

$$\int e^{-x^2} \mathrm{d}x = C + x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!} + \cdots \frac{x^{2n+1}}{(2n+1) \cdot n!} + \cdots$$

Now, lets find what n will give us our desired accuracy. In order to do these we employ our formula for remainder,

$$|R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1} < \frac{M}{(n+1)!} \le \frac{1}{(2n+3) \cdot (n+1)!} < .001.$$

We see that $n = 4$ does the trick. This means that,

$$\int_0^1 e^{-x^2} \mathrm{d}x \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475.$$

(2) $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.
Solution: We employ the Taylor series of e^x, but we only need the first few terms.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots) - 1 - x}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots}{x^2} = \lim_{x \to 0} \frac{1}{2} + \frac{1}{6}x + \cdots = \frac{1}{2}.$$

(3) Find the Taylor series of $e^x \sin x$ up to the first three nonzero terms.
Solution: Note, this is not a recommended technique to solve the problem. Unless you are extremely confident about this you should not use it! The best way would be to just take three derivatives and compute it manually. Regardless, this is a way to solve the problem,

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right) = x + x^2 + \frac{1}{3}x^3 + \cdots .$$

(4) Find the Taylor series of $\tan x$ up to the first three nonzero terms.
Solution: We know the Taylor series of $\sin x$ and $\cos x$, and since $\tan x = \sin x/\cos x$ we can use long division,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + x^5/5! + \cdots}{1 - x^2/2 + x^4/4! + \cdots} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots .$$

(5) Find the Taylor series of $\cosh x$.
Solution: This is pretty easy if we remember the definitions. If we don't remember the definitions, just take a bunch of derivatives as per usual.

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{x^n}{n!} + (-1)^n\frac{x^n}{n!}\right)\right] = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$$

10.10 Binomial Series and More Taylor Series

Consider the series $f(x) = (1+x)^m$, lets calculate the Taylor series. We notice that $f^{(n)}(x) = m(m-1)\cdots(m-(n-1))(1+x)^{m-n}$, and when this is evaluated at $x = 0$ we get $f^{(n)}(x) = m(m-1)\cdots(m-(n-1))$. Then,

$$f(x) = \sum_{n=0}^{\infty}\frac{m(m-1)\cdots(m-(n-1))}{n!} = \sum_{n=0}^{\infty}\frac{m!}{(m-n)!n!}x^n.$$

Technically, the factorials should be replaced with Gamma functions, i.e. $m! = \Gamma(m+1)$, however we're in Calc II, so lets not and say we did.

In statistics we call $\frac{m!}{(m-n)!n!} = \binom{m}{n}$, m choose n. Reminds me of Ralph Wiggum haha - "I choo choo choose you ".

This series is called a binomial series,

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n,\ |x| < 1;\ \binom{m}{n} = \frac{m!}{(m-n)!n!}. \tag{8}$$

Again, to be perfectly accurate those factorials should be Gamma functions.

Lets do an example that we attempted out of the book, but couldn't solve fully because the book put it in the wrong section...

Ex: If we want to find the general formula for $\sqrt{1+x} = (1+x)^{1/2}$. Now, if they ask us to find, say the first three terms always calculate the derivatives!

$$\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n.$$

Something that is mentioned in the book, but something we wont be seeing too much of is Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

Lets look at some examples from the book,

27b) We know what the power series for $1/(1+x^2)$, now we can get $\tan^{-1} x$ by integrating term by term, and again we get the integral of that in a similar fashion. Since we are evaluating these at $x = 0$ and $\tan^{-1}(0) = 0$ and $\int_0^x \tan^{-1} x\big|_{x=0} = 0$, so we need not be concerned with constants of integration.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \Rightarrow \int_0^x \tan^{-1} x \mathrm{d}x = \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$

From this we can find a formula for the Remainder. Notice that since this is an alternating series we use the alternating series remainder because it's more accurate, instead of the Taylor series remainder. Also, notice that our domain is $[0, 1]$.

$$|R_n| \leq \left|\frac{x^{2n+4}}{(2n+3)(2n+4)}\right| \leq \left(\frac{1}{(2n+3)(2n+4)}\right) < 10^{-3} = \frac{1}{1000}.$$

So, we need $n \geq 15$.

27a) This one is fairly easy if you use trial and error but a lot more difficult if you try what we did for the 27b. Basically try it for $n = 0$, it wont work, so try it for $n = 1$, that will work.

47) For this we notice that we can pull out an x^3,

$$x^3 + x^4 + x^5 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{n+3} = x^3 \sum_{n=0}^{\infty} x^n = \frac{x^3}{1-x}.$$

49) Again we'll pull out an x^3, but we'll have to massage it a bit before we can put it into a form that we can use,

$$x^3 - x^5 + x^7 - x^9 + x^{11} + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n+3} = x^3 \sum_{n=0}^{\infty}(-1)^n x^{2n} = x^3 \sum_{n=0}^{\infty}(-x^2)^n = \frac{x^3}{1+x^2}.$$

37) For this, of course you can use L'Hôpital, but the problem asks us to use Taylor series. So, first we compute the Taylor series for $\ln(1+x^2)$ about $x = 0$. Why do we calculate it about $x = 0$. Think about that.

$$L = \lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0} \frac{\ln(1+x^2)}{1-(1-x^2/2+x^4/4!+\cdots)} = \lim_{x\to 0} \frac{\ln(1+x^2)}{x^2/2 - x^4/4! + \cdots}.$$

Notice that the derivative of $\ln(1+x^2)$ is $2x/(1+x^2)$. So we can find the power series of $2x/(1+x^2)$ and integrate term by term,

$$\frac{2x}{1+x^2} = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n} = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1} \Rightarrow \ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}.$$

Again we don't worry about the constant of integration because $\ln(1) = 0$. Then,

$$L = \lim_{x\to 0} \frac{x^2 - x^4/2 + \cdots}{x^2/2 - x^4/4! + \cdots} = \lim_{x\to 0} \frac{1 - x^2/2 + \cdots}{1/2 - x^2/4! + \cdots} = 2.$$