CH. 4 HEAT CONDUCTION

Consider heat conduction in some bulk space V with a boundary ∂V. Also consider an infinitesimal space in that bulk called dV. Let $u(x, y, z, t)$ represent the temperature in V at any time t. Let $E = c\rho u$ where c is the specific heat and ρ is the mass density of the bulk, be the total energy in dV.

There are some fundamental laws that will lead us to the heat equation:

Fourier heat conduction laws:

(1) If the temperature in a region is constant, there is no heat transfer in that region.
(2) Heat always flows from hot to cold.
(3) The greater the difference between temperatures at two points the faster the flow of heat from one point to the other.
(4) The flow of heat is material dependent.

All these laws can be summarized into one equation

$$\phi(x, y, z, t) = -K_0 \nabla u(x, y, z, t) \tag{1}$$

Now we can form a word equation:

$$(\text{Rate of change of heat}) = (\text{Heat flowing into } dV \text{ per unit time}) + (\text{Heat generated in } dV \text{ per unit time}) \tag{2}$$

The first statement is the rate of change of the total energy E. The second is the flux at ∂V in the normal direction. The third is additional heat being generated in dV. For the third statement lets called the additional heat Q. This gives us the equation

$$\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = -\oiint_{\partial V} \phi \cdot n \, dS + \iiint_V Q \, dV \tag{3}$$

And using divergence theorem we get

$$\oiint_{\partial V} \phi \cdot n \, dS = \iiint_V \nabla \cdot \phi \, dV = \iiint_V \nabla \cdot (-K_0 \nabla u) \, dV = K_0 \iiint_V \nabla^2 u \, dV$$

therefore, the equation becomes

$$\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = \iiint_V c\rho \frac{\partial}{\partial t} u \, dV = K_0 \iiint_V \nabla^2 u \, dV + \iiint_V Q \, dV \Rightarrow c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q. \tag{4}$$

If we consider the case $Q = 0$; i.e., no external heat being generated, and if we divide through by $c\rho$, then we get the simplest form of the heat equation

$$\frac{\partial u}{\partial t} = K \nabla^2 u \tag{5}$$

where K is called the thermal diffusivity. In 1-D this is,

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \tag{6}$$

Heat equation examples. Consider the heat equation with a generic initial condition,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = f(x). \tag{7}$$

with the following boundary conditions

Ex: $u(0,t) = u(L,t) = 0$.

Solution: We make the Ansatz, $u(x,t) = T(t)X(x)$. Then we plug this into our heat equation

$$u_t = T'(t)X(x),\ u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}.$$

Since the LHS is a function of t alone, and the RHS is a function of x alone, and since they are equal, they must equal a constant. Lets call it $-\lambda^2$. Then we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \tag{8}$$

Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue λ^2. Now we must solve the two differential equations.

The T equation is the easiest to solve

$$\frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \frac{dT}{dt} = -k\lambda^2 T \Rightarrow \frac{dT}{T} = -k\lambda^2 dt \Rightarrow \int \frac{dT}{T} = \int -k\lambda^2 dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t}$$

Notice that we don't include the constant in front of the exponential, and that is because the X equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don't have to though.

Now, we solve the X equation by recalling our Sturm-Liouville problems

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0.$$

If we look at the $\lambda = 0$ case we have $X(0) = c_2 = 0$ and $X(L) = Lc_1 = 0$, so $X \equiv 0$.

Now we look at the $\lambda \neq 0$ case. $X(0) = A = 0$ and

$$X(L) = X(L) = B\sin\lambda x = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin\frac{n\pi}{L}x \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Next we combine the T and X solutions to get the general solutions,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{9}$$

And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have

$$u(x,0) = \sum_{n=1} B_n \sin\frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Then our full solution is

$$u(x,t) = \frac{2}{L}\sum_{n=1} \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{10}$$

Ex: $u_x(0,t) = u_x(L,t) = 0$.

Solution: We know from the first example that $T = e^{-k\lambda^2 t}$.

For the X equation we need to look at our two cases. For $\lambda = 0$ we have $X = c_1 x + c_2$, and $X'(x) = c_1$, so for both boundaries $X'(0) = c_1 = X'(L)$. These leaves us with a constant $X = c_2$.

For the $\lambda \neq 0$ case we have

$$X = A\cos\lambda x + B\sin\lambda x \Rightarrow X' = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$$

Then we get $X'(0) = \lambda B = 0$ and

$$X'(L) = -\lambda A\sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = A_n\cos\frac{n\pi x}{L} \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Next we combine the T and X solutions to get our general solution

$$u(x,t) = c_2 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \tag{11}$$

Now we find our coefficients by invoking the initial condition and using Fourier Series

$$u(x,0) = c_2 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$$

This gives us

$$c_2 = \frac{1}{L}\int_0^L f(x)dx$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Combining everything we get the full solution

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{12}$$

Ex: Now lets think of heat transfer in a circle. If we go around in one direction we hit $x=-L$ and in the other direction $x=L$, but these are the same point. So we get the following boundary conditions

$$u(-L,t) = u(L,t), u_x(-L,t) = u_x(L,t) \tag{13}$$

Solution: We know from the previous two problems that our solutions will be

$$T = e^{-k\lambda^2 t}$$
$$X = c_1 x + c_2 \text{ for } \lambda = 0$$
$$X = A\cos\lambda x + B\sin\lambda x \text{ for } \lambda \neq 0$$

For $\lambda = 0$, $X(L) = c_1 L + c_2$ and $X(-L) = -c_1 L + c_2$, so $c_1 = 0$. And the derivative is trivially satisfied.

For $\lambda \neq 0$,

$$X(L) = X(-L) \Rightarrow A\cos\lambda L + B\sin\lambda L = A\cos\lambda L - B\sin\lambda L \Rightarrow \sin\lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

And

$$X'(L) = X'(-L) \Rightarrow -\lambda A\sin\lambda L + \lambda B\sin\lambda L = \lambda A\sin\lambda L + \lambda B\cos\lambda L \Rightarrow \sin\lambda L = 0$$

But we already showed this. So, we need to keep both coefficients. Then our solution for X, which as we saw in previous conditions (for the heat equation) is just the initial condition of the general solution, is

$$X = c_2 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L} = u(x,0) = f(x) \tag{14}$$

Now we use Fourier series to solve for the coefficients,

$$c_2 = \frac{1}{L}\int_0^L f(x)dx$$
$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$
$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Putting everything back into the general solution gives us

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx + \sin\frac{n\pi x}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{15}$$

Nonhomogeneous heat conduction examples.

Ex: Consider the following nonhomogeneous boundary condition problem

$$u_t = ku_{xx}; \qquad u(0,t) = A, \quad u(L,t) = B; \qquad u(x,0) = f(x). \tag{16}$$

We first look for the easiest solution: the equilibrium temperature. What does equilibrium mean? We solve the problem

$$\frac{\partial u_*}{\partial t} = 0 \Rightarrow \frac{\partial^2 u_*}{\partial x^2} = 0; \qquad u_*(0) = A, \quad u_*(L) = B.$$

So, $u_* = c_1 x + c_2$, and $u_*(0) = c_2 = A$, $u_*(L) = c_1 L + A = B$, then our equilibrium solution is $u_* = \frac{B-A}{L}x + A$. Obviously, this does not solve the problem, but it does allow us to make a change of variables that makes the B.C.'s homogeneous. Let $v(x,t) = u(x,t) - u_*(x)$. Taking a time derivative kills u_* and taking two spatial derivatives also kills u_*, so we get

$$v_t = kv_{xx}; \qquad v(0,t) = v(L,t) = 0; \qquad v(x,0) = f(x) - u_* = f(x) - \frac{B-A}{L}x + A \tag{17}$$

We know

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \tag{18}$$

then

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x) - \frac{B-A}{L}x + A$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L (f(x) - \frac{B-A}{L}x + A)\sin\frac{n\pi x}{L}dx$$

which gives us

$$u(x,t) = \frac{B-A}{L}x + A + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{19}$$

Ex: Now lets look at an example where the PDE itself is nonhomogeneous

$$u_t = ku_{xx} + Q; \qquad u(0,t) = A, \quad u(L,t) = B; \qquad u(x,0) = f(x). \tag{20}$$

Then $u_{xx} = -Q/k \Rightarrow u_* = -Qx^2/2k + c_1 x + c_2$. Plugging in the BCs gives us $u_*(0) = c_1 = A$ and

$$u_*(L) = -\frac{Q}{2k}L^2 + c_1 L + A = B \Rightarrow c_1 = \frac{1}{L}\left[B - A + \frac{Q}{2k}L^2\right] \Rightarrow u_* = -\frac{Q}{2k}x^2 + \frac{x}{L}\left[B - A + \frac{Q}{2k}L^2\right] + A$$

Letting $v(x,t) = u(x,t) - u_*(x)$ gives us our homogenized equation.

4)

$$u_t = ku_{xx}; \qquad u(0,t) = u_0, \quad u(1,t) = u_1; \qquad u(x,0) = f(x) \tag{21}$$

Solution: $u_{xx} = 0 \Rightarrow u_* = c_1 x + c_2$, so $u_*(0) = c_2 = u_0$ and $u_*(1) = c_1 + u_0 = u_1 \Rightarrow c_1 = u_1 - u_0$, then our equilibrium solution is $u_* = (u_1 - u_0)x + u_0$. Letting $v = u - u_*$ gives us our homogenized equation.