SEC. 2.2 PERTURBATION THEORY (CONTINUED)

Before we start the problem let us write down a few key definitions

Definition 1. We say the functions f and g are asymptotically equivalent if and only if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1,$$

and this is denoted as $f\sim g$.

Notice that this works for $x\to 0$ as well since $x\to 0\Rightarrow 1/x\to\infty$.

Definition 2. We say $f>0$ is much less than g if and only if for all $\varepsilon>0$, $f\le\varepsilon g$.

Recall that $\ddot\theta+k\sin\theta=0$ can be solved exactly if $\theta\ll 1$; i.e., $\sin\theta\sim\theta$. Before we can do any asymptotic analysis we need to nondimensionalize our problem. Since we want to study small angles, let $|\varepsilon|\le 1$ be the max possible angle of the system. Then let $x=\theta/\varepsilon$, so the maximum x can be will always be unity. Our ODE becomes

$$\varepsilon\ddot x+k\sin(\varepsilon x)=0\Rightarrow\ddot x+\frac{k}{\varepsilon}\sin(\varepsilon x)=0;\qquad x(0)=1,\dot x(0)=0.\tag{1}$$

Now let $x=\varepsilon^0x_0+\varepsilon^1x_1+\varepsilon^2x_2+o(\varepsilon^3)$. Further, we need to take the Taylor series of Sine as plugging x in directly will not provide any additional simplification. We write

$$\sin(\varepsilon x)=\varepsilon x-\frac{\varepsilon^3x^3}{6}+o(\varepsilon^5).\tag{2}$$

Then the expanded ODE becomes

$$\varepsilon^0\ddot x_0+\varepsilon^1\ddot x_1+\varepsilon^2\ddot x_2+o(\varepsilon^3)+k\left[\varepsilon_0x_0+\varepsilon_1x_1+\varepsilon_2x_2-\varepsilon^2\frac{x_0^3}{6}+o(\varepsilon^3)\right]=0\tag{3}$$

with initial conditions

$$x_0(0)=1,\dot x_0(0)=0;\qquad x_1(0)=\dot x_1(0)=0;\qquad x_2(0)=\dot x_2(0)=0\tag{4}$$

Now we may solve each successive order.

$o(\varepsilon^0)$: $\ddot x_0+kx_0=0$; $x_0(0)=1$, $\dot x_0(0)=0$, then $x_0=\cos\sqrt{k}t$.

$o(\varepsilon^1)$: $\ddot x_1+kx_1=0$; $x_1(0)=\dot x_1(0)=0$, then $x_1\equiv 0$.

$o(\varepsilon^2)$: $\ddot x_2+kx_2=\dfrac{kx_0^3}{6}=\dfrac{k}{6}\cos^3\sqrt{k}t=\dfrac{k}{24}\cos 3\sqrt{k}t+\dfrac{k}{8}\cos\sqrt{k}t$; $x_2(0)=\dot x_2(0)=0$.

Recall that for these types of problems with nontrivial forcing (i.e., right hand side) we use either the method of undetermined coefficients or variation of parameters. For perturbations it is often better to use undetermined coefficients.

First we write the characteristic solution

$$x_c=A\cos\sqrt{k}t+B\sin\sqrt{k}t.\tag{5}$$

Notice that the forcing $\dfrac{k}{24}\cos 3\sqrt{k}t+\dfrac{k}{8}\cos\sqrt{k}t$ has a $\cos\sqrt{k}t$ term, and hence the corresponding Sine and Cosine terms in the particular solution will have a multiple of t. So, our guess for a particular solution is

$$x_p=A_1t\cos\sqrt{k}t+A_2t\sin\sqrt{k}t+B_1\cos 3\sqrt{k}t+B_2\sin 3\sqrt{k}t.\tag{6}$$

Plugging into the ODE and matching terms gives us $A_1=0$, $B_2=0$, $A_2=\sqrt{k}/16$, $B_1=-1/192$. Then our general solution is

$$x_2=x_c+x_p=A\cos\sqrt{k}t+B\sin\sqrt{k}t+\frac{\sqrt{k}}{16}t\sin\sqrt{k}t-\frac{1}{192}\cos 3\sqrt{k}t.$$

Plugging this into the initial conditions gives us $A=1/192$ and $B=0$. Then our complete solution is

$$x_2=\frac{1}{192}\cos\sqrt{k}t++\frac{\sqrt{k}}{16}t\sin\sqrt{k}t-\frac{1}{192}\cos 3\sqrt{k}t.\tag{7}$$

Then the full expanded solution is

$$x = \cos\sqrt{k}t + \varepsilon^2\left[\frac{1}{192}\cos\sqrt{k}t + \frac{\sqrt{k}}{16}t\sin\sqrt{k}t - \frac{1}{192}\cos 3\sqrt{k}t\right] + o(\varepsilon^3). \tag{8}$$

This seems like a perfectly legitimate solution, however you will notice that for large time this solution blows up because of the $\frac{\sqrt{k}}{16}t\sin\sqrt{k}t$ term. We know that this is unphysical, and therefore this cannot be our solution. We need to modify our method in order to find the correct solution.

We will note that the solution works for small time, so perhaps we should expand t as well. This is called the Poincaré–Lindsted method. It should be noted that this may not work for all ODEs with secular terms, but in applied mathematics we don't allow rigor to stifle progress. If it works; it works!

We introduce a new variable and expand around t

$$\tau = \omega t; \omega = 1 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \cdots \tag{9}$$

and

$$x = \varepsilon^0 x_0(\tau) + \varepsilon^1 x_1(\tau) + \varepsilon^2 x_2(\tau) + o(\varepsilon^3). \tag{10}$$

This will end up changing our derivatives to

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \omega\frac{dx}{d\tau} \Rightarrow \ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{d\tau}\frac{d\tau}{dt} = \frac{d}{d\tau}\left(\omega\frac{dx}{d\tau}\right)\frac{d\tau}{dt} = \omega^2\frac{d^2x}{d\tau^2}$$

Our new ODE becomes

$$\omega^2\frac{d^2x}{d\tau^2} + kx = 0; \qquad x(0) = 1,\ \dot{x}(0) = 0. \tag{11}$$

In the notes I will go straight to the successive orders, but if you need to you should write out the ODE in the expanded form to help you pick out the successive ODEs.

$o(\varepsilon^0)$: $x_0'' + kx_0 = 0$; $x_0(0) = 1$, $x_0(0) = 0$, then $x_0 = \cos\sqrt{k}\tau$.

$o(\varepsilon^1)$: $x_1'' + kx_1 = -2\alpha_1 x_0'' = 2\alpha_1 k\cos\sqrt{k}t$; $x_1(0) = x_1(0) = 0$

Last time we saw that the $\cos\sqrt{k}t$ term in the forcing for the order ε^2 problem produced our secular term in the solution. This time we see it in the order ε problem and we need to get rid of this, so $\alpha_1 = 0$, and hence $x_1 \equiv 0$, just as we would expect since this is what we got last time.

$o(\varepsilon^2)$: $x_2'' + kx_2 = -2\alpha_2 x_0'' + \frac{k}{6}x_0^3 = 2\alpha_2 k\cos\sqrt{k}t + \frac{k}{24}(\cos 3\sqrt{k}t + 3\cos\sqrt{k}t)$; $x_2(0) = x_2(0) = 0$. We wish to get rid of the $\cos\sqrt{k}t$ term, so we let $\alpha_2 = -1/16$. This gives us

$$\tau = \left(1 - \frac{1}{16}\varepsilon^2 + \cdots\right)t \tag{12}$$

Notice that if we invert this we get exactly what is given in the book

$$t = \left(1 + \frac{1}{16}\varepsilon^2 + \cdots\right)\tau \tag{13}$$

Once we have this we can plug everything back in to get our complete solution

$$x = \cos\sqrt{k}\left(1 - \frac{1}{16}\varepsilon^2 + \cdots\right)t + \varepsilon^2\left[\frac{1}{192}\cos\sqrt{k}\left(1 - \frac{1}{16}\varepsilon^2 + \cdots\right)t - \frac{1}{192}\cos 3\sqrt{k}\left(1 - \frac{1}{16}\varepsilon^2 + \cdots\right)t\right]$$

which can be further simplified by taking the Taylor series of Cosine.

Introduction to Chaos

Typing "chaos" into Google yields 240,000,000 results in 0.14 seconds.

Google defines chaos as complete disorder and confusion; behavior so unpredictable as to appear random, owing to great sensitivity to small changes in conditions.

Wikipedia says chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions.

Lets see what some experts have to say about chaos.

Lorenz said, "Chaos: when the present determines the future, but the approximate present does not approximately determine the future." We should keep in mind that he is thinking of climate systems.

Poincaré said, "It may happen that slight variations in initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon."

These are philosophical definitions, but how about a mathematical one? Meiss defines it in his book on differential dynamical systems.

Definition 3. A flow ϕ is chaotic on a compact invariant set $\mathbb{X}$ if ϕ is transitive and exhibits sensitive dependence on $\mathbb{X}$.

How about from a classical mechanics point of view? Taylor writes, "This erratic nonperiodic long-term behavior is one of the defining characteristics of chaos. The other defining characteristic is the phenomenon called sensitivity to initial conditions."

Now, lets get back to some mathematical definitions. Glendinning loosely defines a chaotic solution as aperiodic but bounded and nearby trajectories separate rapidly. And formally defines it as,

Definition 4. A continuous map $f : \mathbb{I} \mapsto \mathbb{I}$ is chaotic if and only if f^n has a horseshoe for some $n \geq 1$.

This definition would be much too technical for us at the moment so we will skip over it for now.

Robinson gives a similar definition as Meiss except for maps, but he says Devaney (1989) gave an explicit definition of a chaotic invariant set in an attempt to clarify the notion of chaos. To our two assumptions he adds the assumption that the periodic points are dense in $\mathbb{Y}$ (an invariant set). Although this last property is satisfied by "uniformly hyperbolic" maps like the quadratic map, it does not seem that this condition is at the heart of the idea that the system is chaotic.

So we see here that even the experts disagree on the definition of chaos. Strogatz puts it nicely, "No definition of the term chaos is universally accepted yet, but almost everyone would agree on the three ingredients used in the following working definition,"

Definition 5. Chaos is aperiodic long-term behavior in a deterministic system that exhibit sensitive dependence on initial conditions.

Lets go back to Meiss's formal definition and dissect it. The only two terms that we may not be familiar with are "sensitive dependence" and "transitivity".

Definition 6. A flow ϕ exhibits sensitive dependence on an invariant set $\mathbb{X}$ if there is a fixed r such that for each $x \in \mathbb{X}$ and any $\varepsilon > 0$, there is a nearby $y \in B_\varepsilon(x) \cup \mathbb{X}$ such that $|\phi_t(x) - \phi_t(y)| > r$ for some $t \geq 0$.

This does not mean that all pairs of nearby points act like this, but we can find points that do.

Definition 7. A flow ϕ is topologically transitive on an invariant set $\mathbb{X}$ if for every pair of nonempty, open sets $\mathbb{U}, \mathbb{V} \in \mathbb{X}$ there is a $t > 0$ such that $\phi_t(\mathbb{U}) \cup \mathbb{V} \neq \emptyset$.

Basically, the flow will wander all over $\mathbb{X}$.

Furthermore, if we transform a chaotic system from one set to another we would like it to still be chaotic. The following theorem outlines the conditions under which this is possible.

Theorem 1. *Suppose $\phi_t : \mathbb{X} \mapsto \mathbb{X}$ and $\psi_t : \mathbb{Y} \mapsto \mathbb{Y}$ are flows, $\mathbb{X}$ and $\mathbb{Y}$ are compact, and ϕ is chaotic on $\mathbb{X}$. Then, if ψ is conjugate to ϕ, it too is chaotic.*

But what type of systems can be chaotic? First lets define an omega limit set.

Definition 8. The ω-limit set of a point x is

$$\omega(x) := \{y|\phi_{t_k}(x) \to y as t_k \to \infty\}. \tag{14}$$

We have a similar definition for α-limit sets.

Now, in 1-D flows we can't have chaos because $\omega(x)$ can only be a fixed point. In 2-D flows we can't have chaos because of the Poincaré-Bendixson theorem.

Theorem 2. *Let D be a simply connected subset of $\mathbb{R}^2$ and ϕ be a flow in D. Suppose that the forward orbits of some $p \in D$ is contained in a compact set and that $\omega(p)$ contains no fixed points. The $\omega(p)$ is a periodic orbit.*