General outline of applied mathematics reasoning:

- Formulate scientific problem in mathematical terms (modeling)
- Analyze the mathematical problem (Dynamical Systems, Asymptotics, Numerics, etc)
- Interpret analysis in scientific terms and compare against empirical data.

Scientific phenomena can be modeled using physical laws (e.g., Newton's laws, conservation laws, Maxwell's equations, etc.)

Often we may bypass this stage by recognizing similarities with common phenomena (e.g. Equilibrium, diffusion, waves, or a combination of them).

## 1.2 Galactic structures

Now lets apply some of these techniques to a phenomena most of us probably haven't seen before.

Lets think of what sort of laws govern these systems. We have Newton's laws of motion and gravitation, relativity, electrodynamics, thermodynamics, etc. But which law is most prevalent at the distances given and temporal accuracy desired? Gravity will play the biggest role, but since we can't possible make precise enough measurements at these distances and time frames, we can throw relativity out. Then the most important law becomes Newton's laws of motion and gravitation.

We are also interested in an averaged view. We want to think about the force a galaxy applies on a point source for various galactic structures. The structure itself is determined by the distribution of stars, which we can write as a generic function of space, time, and velocity: $\Psi(x, y, z;\ u, v, w;\ t)$. Then we can calculate the mass-density $\rho$ by integrating

$$\rho(x, y, z;\ t) = m \iiint_{-\infty}^{\infty} \Psi(x, y, z;\ u, v, w;\ t) du dv dw. \tag{1}$$

Of course, we cannot have infinitely fast stars, but the distribution function takes care of these types of anomalies by attaining a value of zero for this velocities; i.e., $\Psi(x, y, z;\ \infty, \infty, \infty; t) = 0$.

Now that we have the density, we may think about the force the entire galaxy applies to objects. In physics this is done by using field equations, and in this particular case, we want to calculate the gravitational field. But first we need to write down the gravitational potential using Newton's law of gravitation:

$$V(x, y, z;\ t) = -G \iiint \frac{\rho(\bar{x}, \bar{y}, \bar{z};\ t) d\bar{x} d\bar{y} d\bar{z}}{\sqrt{(x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2}}. \tag{2}$$

Using this and the following time derivatives

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v,\ \frac{dz}{dt} = w,\ \frac{du}{dt} = -\frac{\partial V}{\partial x},\ \frac{dv}{dt} = -\frac{\partial V}{\partial y},\ \frac{dw}{dt} = -\frac{\partial V}{\partial z},$$

we can calculate the full equation of motion

$$\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} + v\frac{\partial \Psi}{\partial y} + w\frac{\partial \Psi}{\partial z} = \frac{\partial V}{\partial x}\frac{\partial \Psi}{\partial u} + \frac{\partial V}{\partial y}\frac{\partial \Psi}{\partial v} + \frac{\partial V}{\partial z}\frac{\partial \Psi}{\partial w}. \tag{3}$$

As you may notice, this is a fairly complicated equation. We can simplify this for disc-like galaxies using the assumption that it is a flat disk, and hence gravity is working in only the z-direction. Notice however, that we are not making assumptions "will-nilly"; we are using science to justify our laziness. This is important; we may make any assumptions we want as long as we have a scientific justification.

Back to our problem, if gravity is only working on the z-direction our equation becomes

$$w\frac{\partial \Psi}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial \Psi}{\partial w} = 0. \tag{4}$$

The only other thing we would need is a solution for $V$. Since $V$ is a potential it will satisfy the Poisson equation: $\nabla^2 V = f(z)$. If you haven't seen the Poisson equation before, it is just Laplace's equation with a nonzero right hand side. By Newton's law of gravitation we get $f(z) = 4\pi G\rho(z)$. Further, $\nabla^2 = \partial^2/\partial z^2$ in 1-D. Then

$$\frac{\partial^2 V}{\partial z^2} = 4\pi G\rho(z). \tag{5}$$