Online supplement: Numerical simulations

For numerical simulations, we take the random variables σ_i to be logistically distributed. The logistic distribution has the advantage that one can explicity derive the equilibrium of the voting subgame. For simplicity, we assume that the proposer is uninformed, that is, σ_0 is constant.

The distribution functions are

$$\begin{aligned} F(\sigma|L) &= \frac{Ke^{\beta\sigma}}{1+Ke^{\beta\sigma}}, \\ F(\sigma|H) &= \frac{e^{\beta\sigma}}{1+e^{\beta\sigma}}. \end{aligned}$$

for some positive constants β and $K > 1$. The associated density functions are

$$\begin{aligned} f(\sigma|L) &= \frac{K\beta e^{\beta x}}{(1+Ke^{\beta x})^2} = F(\sigma|L)(1-F(\sigma|L)) \\ f(\sigma|H) &= \frac{\beta e^{\beta x}}{(1+e^{\beta x})^2} = F(\sigma|H)(1-F(\sigma|H)). \end{aligned}$$

We use the following ratios repeatedly:

$$\begin{aligned} \frac{F(\sigma|H)}{F(\sigma|L)} &= K^{-1}\frac{1+Ke^{\beta\sigma}}{1+e^{\beta\sigma}} \\ \frac{1-F(\sigma|H)}{1-F(\sigma|L)} &= \frac{1+Ke^{\beta\sigma}}{1+e^{\beta\sigma}} \\ \frac{f(\sigma|H)}{f(\sigma|L)} &= K^{-1}\left(\frac{1+Ke^{\beta\sigma}}{1+e^{\beta\sigma}}\right)^2. \end{aligned}$$

Note that the likelihood ratio increases from K^{-1} to K.

Proposer and voter preferences

We consider a parameterized version of our debt restructuring example. In the status quo, the voters (creditors) recover 1 in liquidation. The proposer (debtor) offers a fraction x of future cash flows, which may either be H or $L < H$. There is no private values component to either voter or proposer preferences. Thus

$$\Delta^H(x) = xH - 1$$

$$\begin{aligned}
\Delta^L(x) &= xL - 1 \\
V^H(x) &= (1-x)H \\
V^L(x) &= (1-x)L \\
\bar{V}^H &= \bar{V}^L = 0.
\end{aligned}$$

Voting equilibrium

The equilibrium condition for the voting stage is

$$\frac{xH-1}{1-xL}\frac{p^H}{p^L}\frac{f(\sigma|H)}{f(\sigma|L)}\left(\frac{1-F(\sigma|H)}{1-F(\sigma|L)}\right)^{n\alpha-1}\left(\frac{F(\sigma|H)}{F(\sigma|L)}\right)^{n-1-(n\alpha-1)}=1.$$

Substituting in for the distribution and density functions, and writing $Q=\frac{1+Ke^{\beta\sigma}}{1+e^{\beta\sigma}}$, we have

$$\frac{xH-1}{1-xL}\frac{p^H}{p^L}K^{-(n-n\alpha+1)}Q^{n+1}=1.$$

Q varies from 1 to K as σ varies from $-\infty$ to ∞. So this has a solution provided

$$\frac{xH-1}{1-xL}\frac{p^H}{p^L}K^{-(n-n\alpha+1)}<1<\frac{xH-1}{1-xL}\frac{p^H}{p^L}K^{-(n-n\alpha+1)}K^{n+1},$$

i.e.,

$$\frac{xH-1}{1-xL}\frac{p^H}{p^L}\in\left(K^{-n\alpha},K^{n-n\alpha+1}\right),$$

i.e., $x\in(\underline{x}_n,\bar{x}_n)$, where

$$\begin{aligned}
\underline{x}_n &= \frac{p^L K^{-n\alpha}+p^H}{p^L L K^{-n\alpha}+p^H H} \\
\bar{x}_n &= \frac{p^L K^{n-n\alpha+1}+p^H}{p^L L K^{n-n\alpha+1}+p^H H}
\end{aligned}$$

When it exists, the solution is

$$Q=\left(\frac{xH-1}{1-xL}\frac{p^H}{p^L}\right)^{-\frac{1}{n+1}}K^{1-\frac{n}{n+1}\alpha}.$$

Note that $e^{\beta\sigma}=\frac{Q-1}{K-Q}$, so that the probability that each voter votes to accept is given by

$$\begin{aligned}
\Pr(A|H) &= 1-F(\sigma|H)=\frac{K-Q}{K-1}. \\
\Pr(A|L) &= 1-F(\sigma|L)=\frac{1}{Q}\frac{K-Q}{K-1}.
\end{aligned}$$

The probability that the offer is accepted in $\omega = H$ is then

$$P_n^H(x) = \sum_{j=n\alpha}^{n} \binom{n}{j} (1 - F(\sigma|H))^j F(\sigma|H)^{n-j},$$

with a similar expression for $\omega = L$.

In the special case of unanimity ($\alpha = 1$), this reduces to

$$\begin{aligned} P_n^H(x) &= (1 - F(\sigma|H))^n = \left(\frac{K - Q}{K - 1}\right)^n \\ P_n^L(x) &= \frac{1}{Q^n} P_n^H(x). \end{aligned}$$

The limit acceptance probabilities (see Lemma 5) are

$$\begin{aligned} P_\infty^H(x) &= \left(\frac{xH - 1}{1 - xL}\frac{p^H}{p^L}K^{-1}\right)^{\frac{K^{-1}}{1-K^{-1}}} \\ P_\infty^L(x) &= \left(\frac{xH - 1}{1 - xL}\frac{p^H}{p^L}K^{-1}\right) P_\infty^H(x). \end{aligned}$$

Proposer payoff

The proposer chooses the offer x to maximize

$$p^H(1 - x) H P_n^H(x) + p^L(1 - x) L P_n^L(x).$$

Numerical simulation

We are unable to find a convenient closed form solution for the proposer's equilibrium offer, or for the equilibrium acceptance probability. Figure 1 displays the result of numerically simulating the above example for simple majority ($\alpha = 1/2$) and unanimity, for both a relatively small number for voters ($n = 12$) and the limiting case ($n = \infty$). We use $p^H = p^L = 1/2$, $K = 3$ and $H = 2$, and a range of values for $L \in [.5, 1.8]$.

As one can see, there is virtually no difference in equilibrium outcomes between the finite and limit cases for unanimity. For the simple majority case the proposer switches from offering x_H to x_L for somewhat lower values of L in the finite case (compared to the limit case), but the qualitative relation between L and equilibrium outcomes is similar. For both the finite and limit case, the voters prefer unanimity for small values of L, the

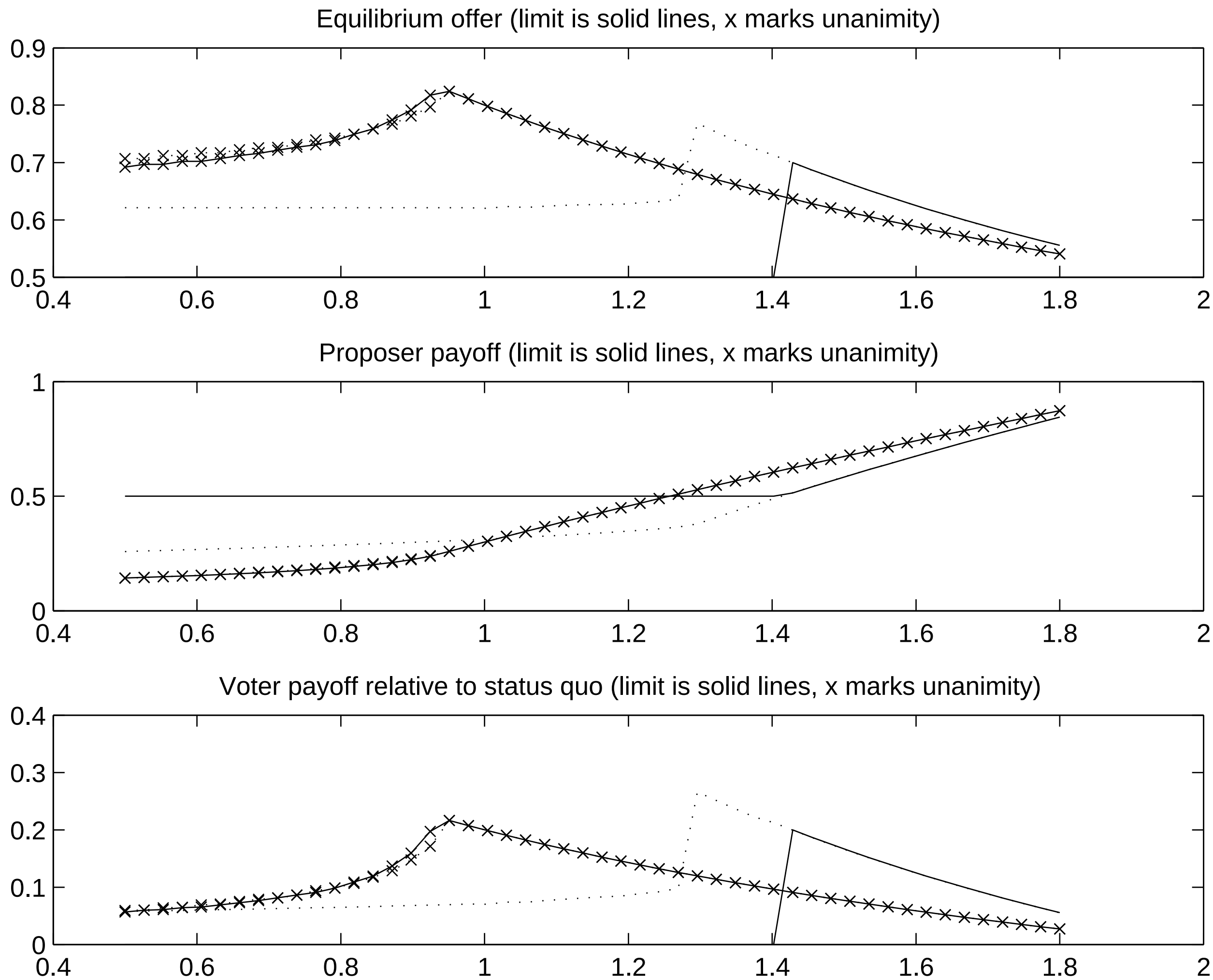

Figure 1: Results of numerical simulation

proposer prefers unanimity for large values of L, and there is an intermediate range of values for which unanimity Pareto dominates.

Online supplement: technical details

Proof of claim in footnote 14

Formally, we show that if $n(1-\alpha)$ voters always reject, the offer is always rejected — that is, a non-responsive rejection equilbrium is played. The proof is by contradiction: suppose to the contary that an equilibrium exists in which $n(1-\alpha)$ voters always reject, but the offer is sometimes accepted. Given Duggan and Martinelli's result, the $n\alpha$ voters behave symetrically, and vote to accept if and only if they observe a signal above σ^* satisfying

$$\Delta^H \left(\frac{1-F(\sigma^*|H)}{1-F(\sigma^*|L)}\right)^{n\alpha-1} \ell(\sigma^*) + \Delta^L \geq 0.$$

Each of the remaining $n(1-\alpha)$ voters is happy to reject for all signal realizations if and only if

$$\Delta^H \left(\frac{1-F(\sigma^*|H)}{1-F(\sigma^*|L)}\right)^{n\alpha-1} \frac{F(\sigma^*|H)}{F(\sigma^*|L)} \ell(\bar{\sigma}) + \Delta^L \leq 0.$$

Hence

$$\ell(\bar{\sigma}) \leq \frac{\ell(\sigma^*)}{\frac{F(\sigma^*|H)}{F(\sigma^*|L)}},$$

which gives the required contradiction since the lefthand side is increasing by assumption, and equals $\ell(\bar{\sigma})$ at $\sigma^* = \bar{\sigma}$.

Derivation of consistency bounds on belief

Conditional seeing σ_0, the proposer assesses the probability of state H as

$$\frac{\Pr(H \text{ and } \sigma_0)}{\Pr(\sigma_0)} = \frac{p^H f_0(\sigma_0|H)}{p^H f_0(\sigma_0|H) + p^L f_0(\sigma_0|L)}.$$

Thus consistency implies that

$$\beta_n(x) \in \left[\frac{p^H f_0(\underline{\sigma}|H)}{p^H f_0(\underline{\sigma}|H) + p^L f_0(\underline{\sigma}|L)}, \frac{p^H f_0(\bar{\sigma}|H)}{p^H f_0(\bar{\sigma}|H) + p^L f_0(\bar{\sigma}|L)}\right],$$

and so

$$\begin{aligned} \frac{\beta_n(x)}{1-\beta_n(x)} &\in \left[\frac{p^H f_0(\underline{\sigma}|H)}{p^L f_0(\underline{\sigma}|H)}, \frac{p^H f_0(\bar{\sigma}|H)}{p^L f_0(\bar{\sigma}|H)}\right] \\ &= \left[\frac{p^H}{p^L}\ell_0(\underline{\sigma}), \frac{p^H}{p^L}\ell_0(\bar{\sigma})\right]. \end{aligned}$$

Proof of Lemma A-1

Rewriting, we must show that

$$\frac{\int^{\sigma} f(\tilde{\sigma}|L)\,\ell(\tilde{\sigma})d\tilde{\sigma}}{\int^{\sigma} f(\tilde{\sigma}|L)\,d\tilde{\sigma}}$$

is increasing in σ. Differentiating, we must show

$$f(\sigma|L)\,\ell(\sigma)\int^{\sigma} f(\tilde{\sigma}|L)\,d\tilde{\sigma} > f(\sigma|L)\int^{\sigma} f(\tilde{\sigma}|L)\,\ell(\tilde{\sigma})d\tilde{\sigma},$$

which is immediate from MLRP. The proof that $(1-F(\sigma|L))/(1-F(\sigma|H))$ is decreasing is exactly parallel; its lower bound follows from l'Hôpital's rule. ∎

Proof of claim in footnote 19

Lemma C-1 ***(Rejection equilibrium)***

Fix belief b, a voting rule $\alpha > \frac{1}{2}+\frac{1}{2n}$ and preferences λ. Let $(\underline{x}_n, \bar{x}_n)$ be the interval defined in Lemma 1. Then if $x \le \underline{x}_n$ the only trembling-hand perfect equilibrium is the non-responsive equilibrium in which each voter always rejects.

Proof: Let Z be as defined in the proof of Lemma 1. Since $x \le \underline{x}_n$, from the proof of Lemma 1

$$b\Delta^H(x,\sigma)\,\ell(\sigma)\left(\frac{F(\sigma|H)}{F(\sigma|L)}\right)^{n-n\alpha}\left(\frac{1-F(\sigma|H)}{1-F(\sigma|L)}\right)^{n\alpha-1}+(1-b)\,\Delta^L(x,\sigma)\le 0$$

for all σ. It follows that

$$b\Delta^H(x,\bar{\sigma})\,\ell(\bar{\sigma})\ell(\bar{\sigma})^{n\alpha-1}+(1-b)\,\Delta^L(x,\bar{\sigma})\le 0.$$

It then follows that for any σ_i,

$$b\Delta^H(x,\sigma_i)\,\ell(\sigma_i)\ell(\bar{\sigma})^{n\alpha-1}+(1-b)\,\Delta^L(x,\sigma_i)\le 0$$

(for this expression could only be strictly positive if $\Delta^H(x,\sigma_i)$ were strictly positive; but then by MLRP it would be strictly positive at $\sigma_i=\bar{\sigma}$). Finally, since $\ell(\bar{\sigma})>1$, for any $m<n\alpha-1$,

$$b\Delta^H(x,\sigma_i)\,\ell(\sigma_i)\ell(\bar{\sigma})^{m}+(1-b)\,\Delta^L(x,\sigma_i)<0.$$

In words, this last inequality says that voter i, having observed his own signal σ_i, will reject the offer x even if he conditions on the event that $m < n\alpha - 1$ other voters observe the most pro-acceptance signal $\bar{\sigma}$. This has two implications.

First, the equilibrium in which all voters reject always is a trembling-hand perfect equilibrium: for if all voters tremble and accept with probability ε independent of their own signal, it remains a best response to reject the offer. This follows since it is a best response to reject the offer given the information that $m < n\alpha - 1$ voters have observed $\bar{\sigma}$, it is certainly a best response to reject given no information.

Second, we claim that the equilibrium in which all voters accept is not trembling-hand perfect. Recall that voter i's vote only matters if exactly $n\alpha - 1$ other voters vote to accept. This event only arises if at least $n - n\alpha$ of the $n - 1$ other voters tremble. As the probability of trembles converges to zero, voter i's best response is determined entirely by the event in which exactly $n - n\alpha$ other voters tremble. But by assumption $n - n\alpha < n\alpha - 1$, and so even if voter i infers from $n - n\alpha$ trembles that $n - n\alpha$ voters have observed $\bar{\sigma}$, his best response is to reject. So the equilibrium in which all creditors accept always cannot be trembling-hand perfect. ∎

Proof of Claim 3 in Lemma 7

Parallel to Claim 1, it suffices to show that $\limsup \sigma_n^* < \sigma_L$. Suppose to the contrary that $\limsup \sigma_n^* \geq \sigma_L$. So for any δ, there exists a subsequence such that $\sigma_n^* \geq \sigma_L - \delta$. By hypothesis, there exists ε such that $x_n \geq x_L + \varepsilon$ for all n large enough. By definition, $\Delta^L(x_L, \sigma_L, \lambda) = 0$; so for δ small enough, there exists $\hat{\varepsilon}$ such that $\Delta^L(x_n, \sigma_n^*, \lambda) > \hat{\varepsilon}$. Moreover, $\Delta^H(x_n, \sigma_n^*, \lambda) \geq \Delta^L(x_n, \sigma_n^*, \lambda)$. Consequently $Z(x_n, \sigma_n^*) > 0$ for n sufficiently large. So σ_n^* cannot be a responsive equilibrium; and since $x_n \geq \underline{x}_n$ it is not a rejection equilibrium either. The only remaining possibility is that σ_n^* is an acceptance equilibrium — but then $\sigma_n^* = \underline{\sigma}$, which gives a contradiction when δ is chosen small enough. ∎

Proof of Claim 4 in Lemma 7

Parallel to Claim 1, it suffices to show that $\limsup \sigma_n^* < \sigma_H$. Suppose to the contrary that $\limsup \sigma_n^* \geq \sigma_H$. So for any $\delta > 0$, there exists a subsequence of σ_n^* such that $\sigma_n^* \geq \sigma_H - \delta$. By hypothesis, there exists ε such that $x_n \geq x_H + \varepsilon$ for all n large enough. By definition $\Delta^H(x_H, \sigma_H, \lambda) = 0$; so for δ small enough, there exists $\hat{\varepsilon}$ such that $\Delta^H(x_n, \sigma_n^*, \lambda) > \hat{\varepsilon}$. Next, define

$$\phi = \min_{\sigma \in [\sigma_H - \delta, \bar{\sigma}]} \frac{(1 - F(\sigma|H))^\alpha F(\sigma|H)^{1-\alpha}}{(1 - F(\sigma|L))^\alpha F(\sigma|L)^{1-\alpha}}$$

Recall that by definition $F(\sigma_H|H) = 1 - \alpha$. By arguments similar to those in Claim 2, it follows that $\phi > 1$ for δ chosen small enough, and so

$$\left(\frac{(1 - F(\sigma^*|H))^\alpha F(\sigma^*|H)^{1-\alpha}}{(1 - F(\sigma^*|L))^\alpha F(\sigma^*|L)^{1-\alpha}} \right)^n \geq \phi^n \to \infty.$$

From Lemma A-1, the term $\frac{1-F(\sigma|L)}{1-F(\sigma|H)}$ lies above $\ell(\bar{\sigma})$. By belief consistency, $\beta_n(x_n)$ is bounded away from zero. Consequently $Z(x_n, \sigma_n^*) > 0$ for n sufficiently large. A contradiction then follows as in Claim 3. ∎

Proof of Lemma A-6, Part 1b

Preliminaries: The first part of the proof consists of defining bounds which we will use to establish uniform convergence below. Choose μ, $\delta_1 \in (0, \delta]$ such that $x_H + \mu < x_L - \mu$, and for all σ_0 for which $W(\sigma_0) < -\varepsilon$,

$$p^H(\sigma_0)(1 - \delta_1)\Delta_0^H(x_H + \mu, \sigma_0) > \delta_1 \max_{\omega, \sigma_0, x} \Delta_0^\omega(x, \sigma_0) \tag{C-1}$$

$$(1 - \delta_1) E\left[\Delta_0^\omega\left(x_L + \frac{\mu}{2}, \sigma_0\right)|\sigma_0\right] > E\left[\Delta_0^\omega(x_L + \mu, \sigma_0)|\sigma_0\right] \tag{C-2}$$

$$\begin{aligned} &(1 - \delta_1) E_\omega\left[\Delta_0^\omega\left(x_L + \frac{\mu}{2}, \sigma_0\right)|\sigma_0\right] \\ > \ &p^H(\sigma_0)\Delta_0^H(x_H - \mu, \sigma_0) + p^L(\sigma_0)\delta_1\Delta_0^L(x_H - \mu, \sigma_0) \end{aligned} \tag{C-3}$$

$$\begin{aligned}&(1-\delta_1)\,E_\omega\left[\Delta_0^\omega\left(x_L+\frac{\mu}{2},\sigma_0\right)|\sigma_0\right]\\>\;&(1-\delta)\,E_\omega\left[\Delta_0^\omega\left(x_L-\mu,\sigma_0\right)|\sigma_0\right]\end{aligned}\tag{C-4}$$

To see that such a choice exists, choose $\mu \leq \delta$ small enough that

$$E_\omega\left[\Delta_0^\omega\left(x_L+\frac{\mu}{2},\sigma_0\right)|\sigma_0\right] > p^H(\sigma_0)\,\Delta_0^H(x_H-\mu,\sigma_0)$$

(this is possible by $W(\sigma_0) < -\varepsilon$) and

$$E_\omega\left[\Delta_0^\omega\left(x_L+\frac{\mu}{2},\sigma_0\right)|\sigma_0\right] > (1-\delta)\,E_\omega\left[\Delta_0^\omega(x_L-\mu,\sigma_0)\,|\sigma_0\right].$$

Then choose δ_1 small enough that (C-1), (C-2), (C-3), (C-4) hold.

Let $\underline{b}$ and $\bar{b}$ respectively denote the most pro-L and pro-H beliefs possible. Fix a realization of σ_0 such that $W(\sigma_0) < -\varepsilon$. Define the following offer sequences, which we use throughout the proof:

$$x_n^{H+} \equiv x_H+\mu, \quad x_n^{H-} \equiv x_H-\mu, \quad , x_n^{L+} \equiv x_L+\frac{\mu}{2}, \quad x_n^{L-} \equiv x_L-\mu.$$

By Lemma 7, $P_n^L\left(x_n^{L+},\underline{b}\right) \to 1$, $P_n^L\left(x_n^{L-},\bar{b}\right) \to 0$, $P_n^H\left(x_n^{H+},\underline{b}\right) \to 1$ and $P_n^\omega\left(x_n^{H-},\bar{b}\right) \to 0$. Thus there exists $N(\varepsilon,\delta)$ such that for $n \geq N(\varepsilon,\delta)$, $P_n^L\left(x_n^{L+},\underline{b}\right) \geq 1-\delta_1$, $P_n^L\left(x_n^{L-},\bar{b}\right) \leq \delta_1$, $P_n^H\left(x_n^{H+},\underline{b}\right) \geq 1-\delta_1$, and $P_n^\omega\left(x_n^{H-},\bar{b}\right) \leq \delta_1$ for $\omega = L, H$.

Fix σ_0 such that $W(\sigma_0) \geq \varepsilon$, along with a sequence of equilibrium offers x_n associated with the realization σ_0.

Part A: $x_n \geq x_H - \mu$ for $n \geq N(\varepsilon,\delta)$.

Proof: Suppose otherwise, i.e., $x_n < x_H - \mu$. So $P_n^L(x_n,b) \leq P_n^H(x_n,b) \leq P_n^H\left(x_H-\mu,\bar{b}\right) \leq \delta_1$. So the proposer's payoff from x_n is bounded above by

$$\delta_1 \max_{\omega,\sigma_0,x} \Delta_0^\omega(x,\sigma_0) + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right].$$

Consider instead the offer $x_H+\mu$. The acceptance probability in state H is at least $1-\delta_1$. Since (by Assumption 5) the proposer is always better off when his offer is accepted, his payoff from this offer is bounded below by

$$p^H(\sigma_0)\,(1-\delta_1)\,\Delta_0^H(x_H+\mu,\sigma_0) + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right].$$

By (C-1), this exceeds his equilibrium payoff, giving a contradiction.

Part B: $|x_n - x_L| \leq \mu \leq \delta$ for $n \geq N(\varepsilon, \delta)$.

Proof: First, suppose that $x_n > x_L + \mu$. Since (by Assumption 5) the proposer is always better off when his offer is accepted, his payoff is bounded above by

$$E\left[\Delta_0^\omega\left(x_L + \mu, \sigma_0\right)|\sigma_0\right] + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right].$$

Consider instead the offer $x_L + \frac{\mu}{2}$. The acceptance probability in state L is at least $1 - \delta_1$, and so his payoff from this offer is bounded below by

$$(1 - \delta_1) E_\omega\left[\Delta_0^\omega\left(x_L + \frac{\mu}{2}, \sigma_0\right)|\sigma_0\right] + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right]. \tag{C-5}$$

By (C-2), this exceeds his equilibrium payoff, giving a contradiction.

Second, suppose that $x_n < x_L - \mu$. The acceptance probability in state L is at most δ_1. From Part A $x_n \geq x_H - \mu$, and so the proposer's payoff is bounded above by

$$p^H(\sigma_0)\,\Delta_0^H(x_H - \mu, \sigma_0) + p^L(\sigma_0)\,\delta_1\Delta_0^L(x_H - \mu, \sigma_0) + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right].$$

Consider instead the offer $x_L + \frac{\mu}{2}$. The acceptance probability in state L is at least $1 - \delta_1$, and so his payoff from this offer is bounded below by expression (C-5). By (C-3), this exceeds his equilibrium payoff, giving a contradiction.

Part C: If $W(\sigma_0) \leq -\varepsilon$ and $n \geq N(\varepsilon, \delta)$, then $P_n^L(x_n) \geq 1 - \delta$. for any equilibrium offer x_n.

Proof: Suppose otherwise, i.e., $P_n^L(x_n) < 1 - \delta$. Given Part B, the proposer's payoff is bounded above by

$$(1 - \delta) E_\omega\left[\Delta_0^\omega(x_L - \mu, \sigma_0)|\sigma_0\right] + E_\omega\left[\bar{V}_0^\omega|\sigma_0\right].$$

As above, the proposer's payoff from the offer $x_L + \frac{\mu}{2}$ is bounded below by expression (C-5). By (C-4), this exceeds his equilibrium payoff, giving a contradiction.

Proof of claim in footnotes 23 and 33

In footnote 33 we observe that there is no pure strategy separating equilibrium if the proposer is fully informed.

To see this, suppose to the contrary that such an equilibrium exists, i.e., that the proposer offers x_H^* when he knows $\omega = H$ and $x_L^* \neq x_H^*$ when he knows $\omega = L$. Since the offer reveals ω, in any pure strategy equilibrium the acceptance probability associated with each offer is either 0 or 1. It cannot be the case that both offers are accepted in equilibrium, since the proposer would always deviate from making the higher offer. The offer x_L^* is only accepted if $x_L \neq \infty$ and $x_L^* \geq x_L$. So if x_L^* is accepted then $x_L^* = x_L$ (since otherwise the proposer would make a lower offer). But then there can be no equilibrium in which x_L^* is accepted but x_H^* is rejected, since the proposer would deviate in state H. Finally, a similar argument implies that there can be no equilibrium in which x_H^* is accepted but x_L^* is rejected.