The equilibrium consequences of indexing*

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First draft: September 2016
This draft: April 2019

Abstract

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting. Individuals incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing. A decline in indexing costs directly increases the prevalence of indexing, thereby reducing the price efficiency of the index and augmenting relative price efficiency. In equilibrium, these changes in price efficiency in turn further increase indexing, and raise the welfare of uninformed traders. For well-informed traders, the share of trading gains stemming from market timing increases relative to stock selection trades.

JEL classification: D82, G14.
Keywords: indexing, welfare.

*We thank seminar audiences at Baruch College, BI Oslo, Boston College, the Hanqing Institute of Renmin University, Hong Kong University, Insead, LBS, LSE, MIT, Michigan State University, Notre Dame, NYU, Stanford, the University of Alberta, UC San Diego, the University of Chicago, the University of Maryland, the University of North Carolina, the University of Rochester, UT Austin, and the University of Virginia and for helpful comments. Any remaining errors are our own.

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1 Introduction

The standard investment recommendation that academic financial economists offer to individual retail investors it to purchase a low-fee index mutual fund or exchange-traded fund (ETF), a strategy often describe as "index investing," or simply "indexing." An increasing number of indexing products are available, and are increasingly inexpensive and accessible, and more and more investors follow this advice.[1] In this paper, we develop a benchmark model to analyze the equilibrium consequences of a decrease in indexing costs, paying particular attention to consequences for participation and welfare.

The direct consequence of a fall in indexing costs is, naturally, to draw investors into indexing, and away from both more active trading strategies and non-participation in financial markets. The marginal investor who switches out of active trading into indexing is one who is relatively uninformed. The same is true for the marginal investor who switches from non-participation to indexing. So the direct consequence of falling indexing costs is to reduce the price efficiency of the index and introduce a common "noise" factor to stock prices of firms covered by the index,[2] while simultaneously reducing "liquidity" in individual stocks.

The central questions we ask in this paper are: What equilibrium effects follow from these direct consequences of a fall in indexing costs? And what is the net effect on investors' welfare? In particular, do equilibrium forces dampen the direct consequences, or are the direct consequences instead self-reinforcing?

We find that the direct consequences are self-reinforcing. Although the reduction in price efficiency associated with indexing may sound undesirable, investor welfare increases, even for the least informed investors, as we discuss below. The increase in the welfare of indexing investors draws in more relatively uninformed investors, amplifying the original effect.

Conversely, the direct consequence of relatively uninformed investors switching from active trading to index investing is to increase the price efficiency of individual stock prices. Although this sounds desirable, it reduces investor welfare. This induces yet more investors to abandon active trading, again amplifying the original effect.

As the above discussion suggests, our key analytical result is that increases in participation in the market for a financial asset increase the welfare of those already participating. Put differently, participation decisions are strategic complements.[3] The underlying economic force is that an individual investor prefers to trade in a market in which the average investor

[1] See, for example, French (2008) and Stambaugh (2014).

[2] See, for example, Ben-David, Franzoni, and Moussawi (2018).

[3] Grossman and Stiglitz (1980) analyze traders' decisions to become informed, taking the set of trading agents as given, and show that information acquisition decisions are strategic substitutes: the incremental trading profits from private information decline as more traders become informed. In contrast, we consider a setting without exogenous noise traders, and analyze traders' participation decisions.

is relatively uninformed; since the marginal investor is less informed than the average investor, this generates strategic complementarity. Although it may seem intuitive that an investor prefers his average counterparty to be uninformed, this same force leads to prices that are more divorced from cash flows, i.e., lower price efficiency, which is often interpreted as being *undesirable.* Concretely, lower price efficiency in our framework means that prices are more exposed to fluctuations in non-financial income that investors wish to partially hedge, i.e., fluctuations in discount rates. As such, establishing an investor's preference for less informed counterparties entails establishing that the benefits of a less informed average counterparty exceed the costs of lower price efficiency.[4] Perhaps surprisingly, and despite the fact that we work with a canonical model of the type introduced by Diamond and Verrecchia (1981), this analysis has not been conducted by the existing literature.

The central empirical predictions of our analysis are about price efficiency. As indexing costs fall, and indexing increases, price efficiency of the index as a whole falls, while the relative price efficiency of individual stocks increases. Moreover, price efficiency is lower for stocks covered by the index than for those outside. As a consequence of these predictions for price efficiency, index reversals become more pronounced, and a greater fraction of the trading profits of relatively informed investors are attributable to "market timing" strategies as opposed to "stock selection" strategies. Section 5 reviews empirical support for these findings.

Asides from its implications for the equilibrium effects of indexing, our paper also contributes to the wider debate of the extent to which the financial sector contributes to social welfare (see, e.g, Baumol (1965)). In particular, we work with a canonical model in which a financial market exists because it facilitates risk-sharing, and show that informed trading generally worsens this risk-sharing function, while uninformed trading improves it. While we believe there is considerable value in isolating the effect of informed trading on a specific function of the financial sector, we also fully acknowledge that our analysis is silent on how informed trading affects other possible functions of the financial sector. For example, we do not speak to the question of whether information produced be financial markets is valuable in incentive contracts or in guiding resource allocation decisions (see Bond, Goldstein, and Edmans (2012) for a survey).

Related literature: In its general theme, our analysis is related to papers such as Subrahmanyam (1991), Cong and Xu (2016), Stambaugh (2014), and Bhattacharya and O'Hara (2017). Subrahmanyam (1991) models the introduction of index futures, while Cong and Xu

[4]This result is related to the so-called "Hirshleifer effect" (Hirshleifer (1971)), but does not follow directly from it; see subsection 3.1 below, and also related discussions in Marín and Rahi (1999) and Dow and Rahi (2003).

(2016) and Bhattacharya and O'Hara (2017) model the introduction of ETFs. An important assumption in all three papers is how the introduction of a new financial product affects the allocation of "noise" or liquidity traders across assets.[5] Stambaugh (2014) analyzes the effects of an exogenous decline in noise traders for financial markets, and in particular, for actively managed investment funds.[6] Relative to all these papers, we model the behavior of all financial market participants, and in particular, analyze how a fall in indexing costs affects participation decisions and welfare.

In an independent, contemporaneous, and complementary paper, Baruch and Zhang (2017) likewise study the equilibrium consequence of indexing, though from a very different perspective. They consider a multi-asset version of Grossman (1976), so that without indexing prices fully reveal agents' private signals. In this setting they show that an exogenous increase in indexing reduces the amount of information prices contain about individual assets, while the amount of information prices contain about aggregates is unaffected.

We have deliberately based our analysis on the canonical model of financial markets of Diamond and Verrecchia (1981). Like Grossman and Stiglitz (1980), Hellwig (1980), and Admati (1985), these authors analyze trade between differentially informed agents, but different from these papers, there are no exogenous "noise" or "liquidity" trades. Instead, agents have heterogeneous and privately observed exposures to risk. Consequently, financial markets hold the potential to increase welfare by allowing agents to redistribute risk. Perhaps surprisingly, and although a model of this type has been analyzed by a significant number of authors, results on welfare are scarce.[7] One significant algebraic complication in characterizing welfare is that, when combined with the asset price, each agent's private exposure shock contains information about expected asset payoffs. To avoid this complication, Verrecchia (1982) and Diamond (1985) both consider a sequence of economies in which the variance of each individual's exposure shock grows with the number of agents,

[5]Subrahmanyam (1991), and Cong and Xu (2016) allow for optimization by a subset of these traders.

[6]Also related is Gârleanu and Pedersen (2018), who among other things consider an investor's choice between actively and passively managed funds. However, they consider a financial market with a single risky asset, making it hard to consider "indexing."

[7]For results on the effect of information on welfare in different equilibrium models of financial markets, see, for example, Schlee (2001) and Kurlat (forthcoming). The former paper analyzes the value of public signals in a setting in which individual endowments are public once they are realized, and so trades can be conditioned on them, and characterizes conditions in which improvements in public information cause a Pareto deterioration. In contrast, in our setting (inherited from Diamond and Verrecchia (1981)), individual signals are private, and individual endowments are likewise private, even after they are realized. The latter paper analyzes what is essentially an origination market in which sellers are strictly better informed than buyers, and characterizes the ratio of the social to private value of buyer information. In this setting, improvements in seller information reduce adverse selection, and so the social value of information is positive. In contrast, in our setting improvements in trader information do not necessarily reduce adverse selection, and our results imply that improvements in information of a positive measure of agents reduce the welfare of all counterparties.

and directly study the limit of this sequence. In the limit economy, each agent's exposure shock has infinite variance, and so expected utility prior to the realization of the exposure shock is undefined, in turn making it impossible to analyze participation decisions prior to the realization of exposure shocks.[8]

In an independent, contemporaneous, and complementary paper, Kawakami (2017) also makes progress in characterizing welfare in a setting along the lines of Diamond and Verrecchia (1981). Whereas we focus on an economy with a continuum of agents and allow for heterogeneity in the precision of signals about cash flows that agents observe, thereby allowing us to consider the effect an increase in participation by relatively uninformed agents, Kawakami instead considers a finite-agent economy with homogeneous signal precisions, in which an increase in the size of the market is associated with better diversification of individual exposure shocks. Analytically, we make more explicit use than Kawakami of market-clearing conditions, which allows us to incorporate heterogeneity in signal precisions in a tractable way.

Marín and Rahi (1999) obtain welfare results in a relatively specialized setting: there are two classes of agents, one class of which sees identical signals about asset payoffs and private endowments, and another class of completely uninformed agents. Moreover, the traded asset is in zero net supply. Dow and Rahi (2003) also analyze welfare, and obtain some tractability by inserting a risk-neutral market maker into the economy, which reduces the applicability of the model for analyzing aggregate financial markets. In closely related settings, Medrano and Vives (2004) argue that "the expressions for the expected utility of a hedger . . . are complicated," whereas Kurlat and Veldkamp (2015) write that "there is no closed-form expression for investor welfare." The complications, common to our model as well, stem from the role of exposure shocks as signals about asset cash flows, on top of the standard risk sharing role that motivates trade.

Finally, we emphasize that we examine "indexing" in the sense of a "passive" investment strategy based on an index. Other authors have analyzed the distinct topic of the role of indices as benchmarks that affect the compensation of fund managers: see, e.g., Admati and Pfleiderer (1997), Basak and Pavlova (2013), Buffa, Vayanos, and Woolley (2014).

[8] If instead one modeled participation decisions as being made after the exposure shock, then almost all agents would participate, since their exposure shocks are so large.

2 The model

2.1 Preferences, assets, endowments, information

We work with a version of Diamond and Verrecchia (1981) in which there are a continuum of agents (see Ganguli and Yang (2009) and Manzano and Vives (2011)), indexed by the unit interval, $i \in [0,1]$, and multiple assets. We emphasize that this is a canonical setting, in which risk-sharing benefits lead to gains from trade, which in turn allows for informed trading.

Each agent i has preferences with constant absolute risk aversion (CARA) over terminal wealth W_i, and a coefficient of absolute risk aversion of γ. There are m risky assets available for trading. Each asset $k \in \{1,\ldots,m\}$ produces a normally distributed payoff $\tilde{X}_k$. The asset payoffs $\left\{\tilde{X}_k\right\}$ are identically and independently distributed, with common variance τ_X^{-1}. The price of asset k is $\tilde{P}_k$, which is determined in equilibrium. We characterize the competitive equilibrium of the economy, where agents are small relative to the market, and act as price-takers.

Each asset k is in positive net supply, where agent i's initial endowment of asset k is given by $\tilde{s}_{ik}$. The aggregate (and hence per capita) endowment of each asset k is $\tilde{S} = \int_0^1 \tilde{s}_{ik} di$, and is equal across different assets. We denote by $\tilde{\theta}_{ik}$ agent i's trade of asset k.

In addition, agents have other sources of income (e.g., labor income, non-traded capital income) that are correlated with the cash flows of the risky assets. For simplicity, we assume the correlation is perfect, and write agent i's income from sources other than the risky asset as

$$\sum_{k=1}^{m} \left(\tilde{Z}_k + \tilde{u}_{ik}\right) \tilde{X}_k. \tag{1}$$

Here, $\tilde{Z}_k + \tilde{u}_{ik}$ represents agent i's non-financial exposure to the cash flow risk $\tilde{X}_k$. Agent i privately observes the sum $\tilde{Z}_k + \tilde{u}_{ik}$, but not its individual components $\tilde{Z}_k$ and $\tilde{u}_{ik}$. So agent i knows his own income exposures $\tilde{Z}_k + \tilde{u}_{ik}$, but remains uncertain about the aggregate component of other agents' exposures, $\tilde{Z}_k$. Both $\tilde{Z}_k$ and $\tilde{u}_{ik}$ are randomly distributed normal variables, which are independent across assets k, and in the case of $\tilde{u}_{ik}$ independent across agents i also. The variances of $\tilde{Z}_k$ and $\tilde{u}_{ik}$ are τ_Z^{-1} and τ_u^{-1} respectively, and $\mathbb{E}\left[\tilde{Z}_k\right] = 0$. We assume throughout that, for all agents i,

$$\tilde{s}_{ik} + \mathbb{E}\left[\tilde{u}_{ik}\right] = \tilde{S}. \tag{2}$$

Hence while some agents may have greater endowments of the financial asset k, and other agents may have more non-financial exposure to cash flow risk $\tilde{X}_k$, the net exposure of all agents is the same. This assumption is important in allowing us to tractably analyze expected utilities and participation decisions; specifically, we use (2) to obtain the expression for expected utility in Proposition 2.

Note that agents' differential and privately observed exposures $\tilde{Z}_k + \tilde{u}_{ik}$ are the source of gains from trade in the financial market. In equilibrium, aggregate exposures $\tilde{Z}_k$ affect prices, and so fluctuations in $\tilde{Z}_k$ are effectively fluctuations in discount rates. It is also worth noting that it is possible to put a more behavioral interpretation on $\tilde{Z}_k + \tilde{u}_{ik}$; looking ahead to agents' optimal trade (15), $\tilde{Z}_k + \tilde{u}_{ik}$ can be interpreting simply as a shock to agent i's desired holding of asset k, independent of the source of this shock.

The terminal wealth of agent i is determined by the combination of trading profits, initial asset endowments $\tilde{s}_{ik}$, and other income (1). For notational convenience, define

$$\tilde{e}_{ik} = \tilde{s}_{ik} + \tilde{Z}_k + \tilde{u}_{ik}$$

to represent agent i's net exposure to cash flow $\tilde{X}_k$, stemming from both his initial holding of the financial asset k, and his non-financial exposure. From (2), it follows that

$$\mathbb{E}\left[\tilde{e}_{ik}|\tilde{Z}_k\right] = \tilde{S} + \tilde{Z}_k. \tag{3}$$

The terminal wealth of an agent who makes the vector of trades $\tilde{\theta}_i$ is

$$W\left(\tilde{\theta}_i, \tilde{e}_i\right) \equiv \sum_{k=1}^{m} \left(\tilde{\theta}_{ik}\left(\tilde{X}_k - \tilde{P}_k\right) + \tilde{s}_{ik}\tilde{X}_{ik} + \left(\tilde{Z}_k + \tilde{u}_{ik}\right)\tilde{X}_k\right) = \sum_{k=1}^{m} \left(\left(\tilde{\theta}_{ik} + \tilde{e}_{ik}\right)\left(\tilde{X}_k - \tilde{P}_k\right) + \tilde{e}_{ik}\tilde{P}_k\right). \tag{4}$$

Prior to trading, each agent i observes private signals of the form

$$\tilde{y}_{ik} \quad = \quad \tilde{X}_k + \tilde{\epsilon}_{ik},$$

where $\tilde{\epsilon}_{ik}$ is normally distributed with mean 0 and variance τ_i^{-1}, and independent across agents and assets.[9]

Note that the precisions of private signals are heterogeneous across agents, so that some agents are more informed than others. Without loss of generality, we order agents so that signal precision τ_i is decreasing in i; and for simplicity, we assume τ_i is strictly decreasing.[10]

[9]Note that an agent i has the same quality signal about all assets. We leave the interaction of heterogeneity of signal precision across assets with heterogeneity of signal precisions across agents for future research.

[10]Formally, our model is one in which all agents either invest directly in individual stocks, or else invest via passive index funds (see below). An alternative interpretation is that agents with a low i index (and hence precise signals) are relatively good at identifying skilled mutual and hedge funds (Gârleanu and Pedersen (2018)), and the direct investments in the model are made through such intermediaries. Looking ahead, and

An agent's information set at the time of trading is hence the triple of m-vectors $\left(\tilde{y}_i, \tilde{e}_i, \tilde{P}\right)$, which consists of his signals about cash flows $\tilde{y}_i$, his own exposure $\tilde{e}_i$, and the price $\tilde{P}$. We remark that, in contrast to models based on noise traders or other random supply (Grossman and Stiglitz, 1980), agents in our model have additional signals that are relevant at the trading stage, namely their observations of exposure shocks $\tilde{e}_i$, on top of the information conveyed by their private signals $\tilde{y}_i$, and by prices $\tilde{P}$.

2.2 Indexing and participation

Agents incur a cost $\kappa > 0$ of fully participating in financial markets, reflecting a combination of information collection and processing costs, psychic costs, expected trading costs, and the cost of potentially trading in a less than optimal way. Agents make participation decisions prior to observing any of $\left(\tilde{y}_i, \tilde{e}_i, \tilde{P}\right)$. This timing assumption for the participation decision is important for tractability, since it ensures that all random variables are normally distributed at the trading stage.

In addition to fully participating in financial markets, agents have the option of participating only via trading an "index" asset. The index covers the first $l \leq m$ of the m assets, where we assume that l is a power of 2 (this greatly enhances tractability, as will become clear in the next subsection). Since all assets have the same supply $\tilde{S}$, equal-weighted and value-weighted indices coincide. The cash flow produced by the index is hence

$$X_1 \equiv l^{-\frac{1}{2}} \sum_{k=1}^{l} \tilde{X}_k, \tag{5}$$

where $l^{\frac{1}{2}}$ is an index divisor, chosen so that $\operatorname{var}(X_1) = \tau_X^{-1}$.

Formally, we denote by $\tilde{\Theta}_l$ the set of trade vectors that are feasible for an indexing agent, i.e., trades in which agent i buys or sells equal amounts of all assets in the index, and zero units of any asset outside the index:

$$\tilde{\Theta}_l = \left\{\tilde{\theta}_i \in \mathbb{R}^m : \tilde{\theta}_{ij} = \tilde{\theta}_{ik} \text{ for any } j, k \in \{1, \ldots, l\} \text{ and } \tilde{\theta}_{ik} = 0 \text{ for } k > l\right\}.$$

The advantage of participating in financial markets only via indexing is that the participation cost is lower, which we denote by $\kappa_1 \in (0, \kappa)$. The lower participation cost of indexing reflects lower trading costs, because of the availability of low cost index mutual

as one would expect, agents' desired trades depend on their exposure realizations. So in the intermediated investment interpretation just described, agents would also pay attention to general "styles" of funds, in addition to the skill of managers. See also García and Vanden (2009).

funds and exchange traded funds (ETFs); lower cognitive demands and attention costs; and lower information costs, since as our formal analysis will show, a sufficient statistic for an agent's private information if he is indexing is the sum of signals related to the assets in the index, $\sum_{k=1}^{l} \tilde{y}_{ik}$, which can be interpreted as agent i simply paying attention to broad economic aggregates, instead of individual stocks.

Looking ahead, the main comparative static we will be interested in is a fall in κ_1, the participation cost associated with indexing. This corresponds to falling fees, greater availability, and greater awareness of products such as low-cost index funds and ETFs. It may also reflect an increase in public awareness of the standard advice given by finance academics.

Finally, if an individual does not participate in financial markets at all, he pays no participation cost, but does not trade, i.e., $\tilde{\theta}_{ik} = 0$ for all assets k.

The definition of an equilibrium in terms of pricing, participation decisions, and trading strategies is standard. However, we postpone a formal statement of equilibrium conditions until subsection 2.4. This allows us to give the definition directly in terms of a spanning set of synthetic assets, which we introduce next, and use to conduct our analysis.

2.3 Disentangling markets

Although the fundamentals of different assets are independent in all dimensions, the presence of indexing agents introduces a link between the prices of distinct assets. For example, if j and k are two distinct assets covered by the index, then an indexing agent's exposure $\tilde{e}_{ij}$ to cash flow risk $\tilde{X}_j$ affects the agent's desired trade of asset k as well as of asset j.

Because of the entanglement that indexing produces between different assets covered by the index, it is analytically very convenient to change basis and study the economy in terms of a set of synthetic assets that are mutually independent even in the presence of indexers.[11]

The case of two assets ($l = m = 2$) is simple. The first synthetic asset is the index portfolio, $X_1 = \frac{1}{\sqrt{2}} \left(\tilde{X}_1 + \tilde{X}_2 \right)$. The second synthetic asset is $X_2 = \frac{1}{\sqrt{2}} \left(\tilde{X}_1 - \tilde{X}_2 \right)$, which can be labeled a "spread" asset, as it allows agents to trade on the relative mispricing between assets 1 and 2. We next generalize this construction to $l > 2$ assets in the index. We first give a simple example, and then we formalize our change of basis.

Example: Suppose there are $m = 5$ assets and the index covers the first $l = 4$. Then consider the following set of 5 synthetic assets, where the 1st synthetic asset pays X_1 as defined in (5); the 5th synthetic asset coincides with the underlying asset 5, i.e., it pays

[11]This change of basis is not essential to solve for equilibrium prices; see Admati (1985)'s analysis of a multi-asset version of Hellwig (1980).

$X_5 = \tilde{X}_5$; and the remaining 3 synthetic assets pay X_2, X_3, X_4 defined by

$$\begin{aligned} X_2 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_3 - \tilde{X}_2 - \tilde{X}_4\right), \\ X_3 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_2 - \tilde{X}_3 - \tilde{X}_4\right), \\ X_4 &= \frac{1}{2}\left(\tilde{X}_1 + \tilde{X}_4 - \tilde{X}_2 - \tilde{X}_3\right). \end{aligned}$$

The five synthetic assets span the underlying assets $\tilde{X}_k$. Moreover, they are uncorrelated (i.e., $\text{cov}(X_j, X_k) = 0$ for all j, k), and each has variance τ_X^{-1}. The first synthetic asset is simply the index, while synthetic assets 2, 3, and 4 constitute three different long-short trades of assets within the index.

We construct synthetic assets in a way that generalizes the properties in the example. In particular, the first synthetic asset corresponds to the index, while synthetic assets $2, \ldots, l$ correspond to long-short trades of assets contained in the index. Mathematically, this generalization follows from the following straightforward result in matrix algebra:

Lemma 1 *For any positive integers m and $l \leq m$ such that l is a power of 2, there exists an $m \times m$ matrix A with the following properties: A is symmetric and invertible, with $A^{-1} = A$ (i.e., A is involutory); $A_{jk} = 0$ if $j \neq k$ and either $j > l$ or $k > l$; $A_{jk} = 1$ if $j = k > l$; $A_{1k} = l^{-\frac{1}{2}}$ and $|A_{jk}| = l^{-\frac{1}{2}}$ for all $j, k \leq l$; for any j, $j' \neq j$, $\sum_{k=1}^{m} A_{jk}A_{j'k} = 0$; and $\sum_{k=1}^{l} A_{jk} = 0$ for $j = 2, \ldots, l$.*

Given the existence of a matrix A of the type established in Lemma 1, we define synthetic assets $1, \ldots, m$ as paying off

$$X_k \equiv \sum_{j=1}^{m} A_{kj}\tilde{X}_j. \tag{6}$$

In matrix form, the vector of cash flows produced by synthetic assets is hence $X = A\tilde{X}$, which is equivalent to $\tilde{X} = AX$, where both here and below we use the Lemma 1 property that $A = A^\top = A^{-1}$. As in the example, the synthetic assets are uncorrelated, and each has variance τ_X^{-1}. The mean of synthetic asset X_1 is $\sqrt{l}\mathbb{E}\left[\tilde{X}_1\right]$, the mean of assets $2, \ldots, l$ is 0, and the mean of assets $k > l$ is simply $\mathbb{E}\left[\tilde{X}_k\right]$.

For all exogenous variables, we use tildes to denote the underlying fundamental quantity, and the absence of a tilde to denote a variable constructed analogously to (6). For example, we define $Z_k \equiv \sum_{j=1}^{m} A_{kj}\tilde{Z}_j$. From (3), $\mathbb{E}\left[e_{i1}|Z_1\right] = \sqrt{l}\tilde{S} + Z_1$; $\mathbb{E}\left[e_{ik}|Z_k\right] = Z_k$ for $k = 2, \ldots, l$; and $\mathbb{E}\left[e_{ik}|Z_k\right] = \tilde{S} + Z_k$ for $k > l$. Accordingly, define $S_1 = \sqrt{l}\tilde{S}$, $S_k = 0$ for $k = 2, \ldots, l$; and $S_k = \tilde{S}$ for $k > l$. Hence $\mathbb{E}\left[e_{ik}|Z_k\right] = S_k + Z_k$ for any synthetic asset k.

Let $\tilde{\theta}_i$ denote the m-vector of agent i's trades of the underlying assets. This delivers income $\tilde{X}^\top \tilde{\theta}_i$ to agent i. Since A is symmetric, $\tilde{X}^\top \tilde{\theta}_i = X^\top A \tilde{\theta}_i$. So the corresponding trade in synthetic assets is $\theta_i = A\tilde{\theta}_i$, which implies $\tilde{\theta}_i = A\theta_i$. Similarly, let $\tilde{P}$ denote the m-vector of prices of the underlying assets. So a trade $\tilde{\theta}_i$ of the underlying assets costs $\tilde{P}^\top \tilde{\theta}_i$, which equals $\left(A\tilde{P}\right)^\top \theta_i$. So the price vectors P and $\tilde{P}$ are related by $P = A\tilde{P}$ and $\tilde{P} = AP$. Finally, the terminal wealth of agent i is (in matrix form) is $\left(\tilde{X} - \tilde{P}\right)^\top \left(\tilde{\theta}_i + \tilde{e}_i\right) + \tilde{P}^\top \tilde{e}_i$, which gives

$$W(\theta_i, e_i) = (X - P)^\top (\theta_i + e_i) + P^\top e_i.$$

Consequently, to solve for equilibrium prices and welfare, we work directly with the synthetic assets described above. By construction, the synthetic assets are independent of each other in all respects. Moreover, and importantly, indexing corresponds simply to the constraint that an agent can trade only synthetic asset 1, with no trade of any of the other synthetic assets. This means we can analyze the equilibrium in the market for each synthetic asset in isolation (given CARA preferences). Formally, we denote the set of trades available to agents who pay the reduced participation cost κ_1 by $\Theta_1 \equiv \{\theta_i \in \mathbb{R}^m : \theta_{ij} = 0 \text{ if } j \neq 1\}$.

2.4 Equilibrium

The equilibrium definition is a straightforward extension of that used in competitive rational expectations models (Grossman and Stiglitz (1980), Hellwig (1980)), with the participation decision incorporated. To ease the formal statement of the participation decision, we start by defining the expected utilities $\mathcal{U}_i^A(e_i)$, $\mathcal{U}_i^I(e_i)$, $\mathcal{U}_i^0(e_i)$ associated with full participation, or "active" trading; with indexing, or "passive" investing; and with non-participation. For consistency with notation later in the paper, we define these objects conditional on the vector of exposure realizations, e_i, and exclusive of the participation costs κ and κ_1. In particular, it is unnecessary for our analysis to explicitly integrate out uncertainty over exposures e_i.

$$\begin{aligned}
\mathcal{U}_i^A(e_i) &\equiv \mathbb{E}\left[\max_{\theta_i} \mathbb{E}\left[u\left(W(\theta_i, e_i)\right)|y_i, e_i, P\right]|e_i\right], \\
\mathcal{U}_i^I(e_i) &\equiv \mathbb{E}\left[\max_{\theta_i \in \Theta_1} \mathbb{E}\left[u\left(W(\theta_i, e_i)\right)|y_i, e_i, P\right]|e_i\right], \\
\mathcal{U}_i^0(e_i) &\equiv \mathbb{E}\left[u\left(\sum_{k=1}^m e_{ik} X_k\right)|e_i\right].
\end{aligned}$$

Definition 1 *A rational expectations equilibrium consists of non-overlapping sets of agents who fully participate, N, and who index, N_1; trading strategies $\{\theta_i(y_i, e_i, P)\}_{i\in[0,1]}$; and* *price functions $P(X, Z)$. The equilibrium conditions are that markets clear almost surely,*

$$\int_0^1 \theta_i\left(y_i, e_i, P\right) di = 0; \tag{7}$$

each agent i's trading strategy is optimal given his participation decision and prices,

$$\begin{aligned}
\theta_i\left(y_i, e_i, P\right) &\in \arg\max_{\hat{\theta}_i} \mathbb{E}\left[u\left(W\left(\hat{\theta}_i, e_i\right)\right)|y_i, e_i, P\right] \ \textit{if } i \in N, && (8)\\
\theta_i\left(y_i, e_i, P\right) &\in \arg\max_{\hat{\theta}_i \in \Theta_1} \mathbb{E}\left[u\left(W\left(\hat{\theta}_i, e_i\right)\right)|y_i, e_i, P\right] \ \textit{if } i \in N_1, && (9)\\
\theta_{ik}\left(y_i, e_i, P\right) &= 0 \textit{ for all assets } k \textit{ if } i \notin N \cup N_1; && (10)
\end{aligned}$$

and participation decisions are optimal, i.e.,

$$\begin{aligned}
\mathbb{E}\left[\mathcal{U}_i^A\left(e_i\right)\exp\left(\gamma\kappa\right)\right] &\geq \max\left\{\mathbb{E}\left[\mathcal{U}_i^I\left(e_i\right)\exp\left(\gamma\kappa_1\right)\right], \mathbb{E}\left[\mathcal{U}_i^0\left(e_i\right)\right]\right\} \ \textit{if } i \in N,\\
\mathbb{E}\left[\mathcal{U}_i^I\left(e_i\right)\exp\left(\gamma\kappa_1\right)\right] &\geq \max\left\{\mathbb{E}\left[\mathcal{U}_i^A\left(e_i\right)\exp\left(\gamma\kappa\right)\right], \mathbb{E}\left[\mathcal{U}_i^0\left(e_i\right)\right]\right\} \ \textit{if } i \in N_1,\\
\mathbb{E}\left[\mathcal{U}_i^0\left(e_i\right)\right] &\geq \max\left\{\mathbb{E}\left[\mathcal{U}_i^A\left(e_i\right)\exp\left(\gamma\kappa\right)\right], \mathbb{E}\left[\mathcal{U}_i^I\left(e_i\right)\exp\left(\gamma\kappa_1\right)\right]\right\} \ \textit{if } i \notin N \cup N_1.
\end{aligned}$$

Throughout, we assume

$$\begin{aligned}
4\gamma^2(\tau_Z^{-1} + \tau_u^{-1}) &< \tau_X && (11)\\
\gamma^2 &> 4\tau_0\tau_u, && (12)
\end{aligned}$$

where τ_0 is the precision of agent 0's information, i.e., the highest precision in the population of agents. Condition (11) ensures that expected utility is well-defined for non-participating agents; without this condition, such agents are exposed to so much risk that their expected utilities are infinitely low. Condition (12) ensures that an equilibrium exists at the trading stage (see Proposition 1 below). Loosely speaking, without this condition there is too much trading based on information relative to trading based on risk-sharing; Ganguli and Yang (2009) impose essentially the same condition.[12]

[12] The main extension in Manzano and Vives (2011) relative to Ganguli and Yang (2009) is that they allow for the error terms in the trader's signals to be correlated. Non-zero correlation eliminates the need for condition (12). Since our focus is on welfare, we choose to study the slightly more tractable model with conditionally independent estimation errors.

3 Informed trading and welfare in each asset market

Given our construction of synthetic assets as being independent from each other in all respects, in this section we analyze the equilibrium of the market for the kth synthetic asset in isolation, where $1 \leq k \leq m$. Likewise, we evaluate the expected utility associated with trading asset k in isolation.

Intuitively, one would expect agents with more precise private information (i.e., low τ_i) to gain more from participation. Accordingly, we conjecture, and later verify, that the set of agents participating in each market k is $[0, n_k]$ for some n_k.

For clarity, we retain the asset subscript k in the main text, while generally omitting it in proofs in the appendix.

3.1 Welfare benchmarks

In order to gain some intuition regarding our welfare results, it is useful to consider a couple of welfare benchmarks. First, in the (symmetric)[13] unconstrained solution to the social planner's problem each agent i has terminal wealth associated with X_k of

$$W_{ik} = (S_k + Z_k) X_k. \tag{13}$$

That is, the aggregate endowment $(S_k + Z_k) X_k$ is simply split equally among agents. This is the outcome that would be obtained if agents could pool risk before knowing their exposures e_{ik}, and if contracts could be written contingent on the realizations of e_{ik}.

A second useful benchmark is the case in which the private signals y_{ik} are replaced with a finite number of public signals about X_k, with all other aspects the same as in the model described above (in particular, exposures e_{ik} are private information, and trade occurs only after agents observe these exposures). In this case, all agents have the same posterior of X_k at the trading stage, and so by market clearing (7) and the expression for the optimal trade θ_{ik} in (15) below, we have that

$$\theta_{ik} + e_{ik} = S_k + Z_k.$$

So each agent's terminal wealth is

$$W_{ik} = e_{ik} P_k + (S_k + Z_k)(X_k - P_k) = (u_{ik} + s_{ik} - S_k) P_k + (S_k + Z_k) X_k. \tag{14}$$

Comparing (13) and (14), each agent is exposed to an additional risk term, $(u_{ik} + s_{ik} - S_k) P_k$

[13] In non-symmetric solutions, each agent has terminal wealth $W_i = (S_k + Z_k) X_k + K_i$, where K_i is a constant, and $\int K_i di = 0$.

in this second benchmark.

The comparison of these two benchmarks illustrates the challenge of characterizing how information about X_k affects welfare. In particular, one can see that ceteris paribus agents prefer the price P_k to have low variance. In turn, P_k has low variance if it is relatively unaffected by both the realization of the cash flow X_k and the aggregate exposure Z_k. But higher-precision signals about X_k may increase P_k's dependence on X_k (increasing the variance of P_k, and corresponding to the Hirshleifer effect) but decrease P_k's dependence on Z_k (reducing the variance of P_k), so that the overall effect is unclear.[14]

3.2 Basic equilibrium properties

As standard in the literature, to characterize an equilibrium we first conjecture key equilibrium characteristics, and then verify that an equilibrium with these characteristics indeed exists. More concretely, we characterize linear equilibria, where the price is a linear function of the cash flow X_k and the aggregate exposure Z_k, in which all agents with sufficiently precise signals participate. In such equilibria, there is a cutoff agent n_k such that all agents $i \leq n_k$ participate, and agents $i > n_k$ do not participate. In this subsection we establish some key equilibrium properties that hold in any equilibrium of this type. While some of these properties are well-known from prior analyses of related economies, to the best of our knowledge Lemma 2, Corollary 1 and Lemma 6 are new. To maximize transparency, we establish these properties as directly as possible, making use primarily of the market clearing condition (7).

In a linear equilibrium, each agent i's optimal trade has the standard mean-variance form,

$$\theta_{ik} + e_{ik} = \frac{1}{\gamma} \frac{\mathbb{E}\left[X_k - P_k | y_{ik}, e_{ik}, P_k\right]}{\operatorname{var}\left(X_k - P_k | y_{ik}, e_{ik}, P_k\right)} = \frac{1}{\gamma} \frac{\mathbb{E}\left[X_k | y_{ik}, e_{ik}, P_k\right] - P_k}{\operatorname{var}\left(X_k | y_{ik}, e_{ik}, P_k\right)}. \tag{15}$$

The form of the optimal trade (15) indicates that the reciprocal of conditional variance, $\operatorname{var}\left(X_k | y_{ik}, e_{ik}, P_k\right)^{-1}$, is an important quantity. Lemma 2 relates the average reciprocal of conditional variance in the economy to the equilibrium covariance between returns $X_k - P_k$ and cash flows X_k. This relationship turns out to be crucial for establishing our central welfare results in Propositions 2 and 3. We stress that the proof is very concise, and makes

[14]It is worth noting that the limiting case of perfect information about X_k is straightforward. In this case, the price P_k simply equals X_k, and so (14) reduces to $W_{ik} = e_{ik} X_k$, which is the autarchy outcome. Hence welfare is minimized by perfect information about X_k, since in this case the financial market cannot provide any risk sharing (the Hirshleifer effect). Our analysis below concerns the more relevant non-limit case. Moreover, note that Diamond (1985) characterizes how welfare changes as the precision of public information changes, though with the mathematical compromises discussed earlier. In Appendix B, we show that welfare in this benchmark case indeed monotonically declines in the precision of public information, though the proof is non-trivial, consistent with the discussion in the main text.

use only of the market clearing condition (7), the general form of demand (15), and the assumption that random variables are distributed normally. As such the result holds in a large class of economies. Nonetheless, we are unaware of a statement of this result in the existing literature.

Lemma 2 *In a linear equilibrium,*

$$\frac{1}{n_k}\int_0^{n_k} \frac{1}{\mathrm{var}\left(X_k|y_{ik},e_{ik},P_k\right)}di = \frac{1}{\mathrm{cov}\left(X_k-P_k,X_k\right)}. \tag{16}$$

Note that Lemma 2 nests the special case in which all agents are completely uninformed about the cash flow X_k, so that the price is unrelated to X_k, and so for any agent i, $\mathrm{var}\left(X_k|y_{ik},e_{ik},P_k\right) = \mathrm{var}\left(X_k\right) = \mathrm{cov}\left(X_k-P_k,X_k\right)$.

Among other things, we use Lemma 2 to characterize the equilibrium risk premium $\mathbb{E}\left[X_k-P_k\right]$. Taking the unconditional expectation of (15) gives

$$\mathbb{E}\left[\theta_{ik}\right]+S_k = \frac{1}{\gamma}\frac{\mathbb{E}\left[X_k-P_k\right]}{\mathrm{var}\left(X_k|y_{ik},e_{ik},P_k\right)}.$$

Combined with market clearing (7) (specifically, $\int_0^{n_k}\mathbb{E}\left[\theta_{ik}\right]di=0$), we obtain:

Corollary 1 *In a linear equilibrium,*

$$\mathbb{E}\left[X_k-P_k\right] = \gamma S_k \mathrm{cov}\left(X_k-P_k,X_k\right).$$

As for Lemma 2, it may help to note that Corollary 1 nests the special case in which no agent has any information, and so $\mathbb{E}[X_k-P_k]=\gamma S_k \mathrm{var}(X_k)$.

Both prices and exposure shocks play two distinct roles in determining an agent's demand: they directly affect demand, and separately, they also affect an agent's beliefs about the cash flow X, thereby indirectly affecting demand. To clarify this dual role, we will write $\theta_{ik}\left(y_{ik},e_{ik},\hat{e}_{ik},P_k,\hat{P}_k\right)$ for the demand of an agent who has exposure e_{ik} and can trade at price P_k, but who evaluates his conditional distribution over X_k using the the information set $\left(y_{ik},\hat{e}_{ik},\hat{P}_k\right)$. Even though $\hat{e}_{ik}=e_{ik}$ and $\hat{P}_k=P_k$, keeping separate track of the two roles of prices and exposure shocks is conceptually useful.

As typical for this class of models, an important equilibrium quantity is the relative sensitivity of price P_k to the true cash flow X_k and aggregate exposure Z_k, which we denote by ρ_k:

$$\rho_k \equiv -\frac{\frac{\partial P_k}{\partial X_k}}{\frac{\partial P_k}{\partial Z_k}}.$$

We refer to ρ_k as the *price efficiency* of the risky asset, since

$$\begin{aligned} \operatorname{var}(X_k|P_k)^{-1} &= \tau_X + \rho_k^2 \tau_Z, & (17)\\ \operatorname{var}(X_k|y_{ik}, e_{ik}, P_k)^{-1} &= \tau_X + \rho_k^2(\tau_Z + \tau_u) + \tau_i. & (18)\end{aligned}$$

These expressions (derived in the proof of Lemma 4) measure the ability of an outside observer and agent i, respectively, to forecast the cash flow X_k.

Price efficiency ρ_k is sufficiently central to our analysis that, before proceeding further, we establish:

Lemma 3 *In a linear equilibrium, $\frac{\partial P_k}{\partial Z_k} \neq 0$, and hence ρ_k is well-defined.*

The following properties of individual demand follow only from Bayesian updating. They hold whenever price P_k is a linear function of X_k and Z_k, regardless of whether P_k is an equilibrium price.

Lemma 4 *If P_k is a linear function of X_k and Z_k then the effects of non-informational factors on demand θ_{ik} are given by*

$$\frac{\partial \theta_{ik}}{\partial e_{ik}} = -1; \quad \frac{\partial \theta_{ik}}{\partial P_k} = -\frac{1}{\gamma}\frac{1}{\operatorname{var}(X_k|y_{ik}, e_{ik}, P_k)}; \tag{19}$$

while the effects of informational factors on demand θ_{ik} satisfy

$$\frac{\partial \theta_{ik}}{\partial y_{ik}} = \frac{\tau_{ik}}{\gamma}; \quad \frac{\partial \theta_{ik}}{\partial \hat{e}_{ik}} = \frac{\rho_k}{\gamma}\tau_u; \quad \frac{\partial P_k}{\partial Z_k}\frac{\partial \theta_{ik}}{\partial \hat{P}_k} = -\frac{\rho_k}{\gamma}(\tau_Z + \tau_u); \quad \frac{\partial P_k}{\partial X_k}\frac{\partial \theta_{ik}}{\partial \hat{P}_k} = \frac{\rho_k^2}{\gamma}(\tau_Z + \tau_u). \tag{20}$$

Furthermore, these imply that

$$\begin{aligned} \frac{\frac{\partial P_k}{\partial Z_k}\frac{\partial \theta_{ik}}{\partial \hat{P}_k}}{\frac{\partial \theta_{ik}}{\partial \hat{e}_{ik}}} &= -\frac{\tau_Z + \tau_u}{\tau_u}, & (21)\\ \frac{\partial P_k}{\partial Z_k}\frac{\partial \theta_{ik}}{\partial \hat{P}_k} + \frac{\partial \theta_{ik}}{\partial \hat{e}_{ik}} &= -\frac{\rho_k}{\gamma}\tau_Z. & (22)\end{aligned}$$

Equalities (19) and (20) are completely standard. As shown by (19), the direct effects of a higher exposure shock and a higher price are negative. The exposure shock actually moves one-to-one with the trading strategy, as agents offset their exposure with their trading. On the other hand, in equilibrium $\frac{\partial P_k}{\partial Z_k} < 0$ and $\rho_k \geq 0$ (see Lemma 5 immediately below), and so the informational factors go in the opposite direction: both a higher exposure shock and a higher price lead agents to update their beliefs above the underlying asset in a positive way, increasing their demand, as shown in (20).

The relative size of the informational effects of e_{ik} and P_k is important for our analysis, and is characterized by (21) and (22). In particular, (22) drives Lemma 6, which in turn is important for strategic complementarity of participation decisions. As noted, in equilibrium $\frac{\partial P_k}{\partial Z_k} < 0$ and $\rho_k \geq 0$. So a higher Z_k is associated with lower prices, which in turn are associated with lower estimates of X_k. Holding u_{ik} fixed, a higher Z_k also leads to a higher value of e_{ik}, thereby raising an agent's estimate of Z_k, and hence (given equilibrium prices) an agent's estimate of X_k. Equation (22) shows that the first of these effects dominates. Intuitively, this is because the exposure shock e_{ik} contains information about future cash flows only because it allows agents to better interpret the information in prices.

We next establish the basic result that aggregate demand for the risky asset is decreasing in the price. Because of the informational content of prices, this is not completely obvious. At the same time, we show that the price is increasing in the asset's payoff, and the price is decreasing in the aggregate exposure shock. We highlight that the proof of this result makes use only of the market clearing condition (7), along with the *signs* (but not magnitudes) established in Lemma 4.

Lemma 5 *In a linear equilibrium, the aggregate demand curve slopes down, i.e.,*

$$\int_0^{n_k} \frac{\partial \theta_{ik}}{\partial P_k} di + \int_0^{n_k} \frac{\partial \theta_{ik}}{\partial \hat{P}_k} di < 0, \tag{23}$$

and the price is an increasing function of X_k, and a strictly decreasing function of Z_k,

$$\frac{\partial P_k}{\partial X_k} \geq 0 \text{ and } \frac{\partial P_k}{\partial Z_k} < 0,$$

and so in particular $\rho_k \geq 0$.

As noted above, knowledge of individual exposure e_{ik} contains information about X_k only because it helps agent i interpret the price (for example, it provides information about whether a high price is due to a high cash flow X_k or a low aggregate exposure Z_k). Because the information in exposure is subsidiary to the information in prices, it is intuitive that prices contain more information than exposures, as formalized in the following result:

Lemma 6 *In a linear equilibrium, the ratio of the informational to non-informational effect of prices on demand exceeds the ratio of the informational to non-informational effect of exposures on demand,*

$$\frac{\left|\int_0^{n_k} \frac{\partial \theta_{ik}}{\partial \hat{P}_k} di\right|}{\left|\int_0^{n_k} \frac{\partial \theta_{ik}}{\partial P_k} di\right|} > \frac{\left|\int_0^{n_k} \frac{\partial \theta_{ik}}{\partial \hat{e}_{ik}} di\right|}{\left|\int_0^{n_k} \frac{\partial \theta_{ik}}{\partial e_{ik}} di\right|}. \tag{24}$$

Lemma 6 turns out to be critical to establishing that participation decisions exhibit strategic complementarity (Proposition 3). We again highlight that its proof makes use only of basic equilibrium properties established above, along with the *signs* (but not magnitudes) established in Lemma 4.

3.3 Equilibrium at the trading stage

The trading stage of our model is almost exactly as in Ganguli and Yang (2009) and Manzano and Vives (2011), with the only difference being that agents in our model have heterogeneous signal precisions. As such, our next result represents a minor extension of these previous papers, to characterize price efficiency under heterogeneous signal precision. Note that, as in these previous analyses, our trading stage features two equilibria; we follow Manzano and Vives (2011) and focus on the stable equilibrium, which is the one with lower price efficiency.[15]

Proposition 1 *Given participation n_k, there is unique stable linear equilibrium, in which price efficiency is $\rho_k = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{1}{\tau_u}\frac{1}{n_k}\int_0^{n_k}\tau_i di}$. Price efficiency ρ_k is decreasing in participation n_k.*

From Proposition 1, price efficiency is determined by the average information precision of agents who actively trade, given by the term $\frac{1}{n_k}\int_0^{n_k}\tau_i di$. As participation increases, newly participating agents lower this average. Even though these agents bring more information to the market, they also bring more trade motivated by risk-sharing concerns, which function in the same way as noise. The hedging needs of the marginal participating agents, who possess lower quality signals, lead the financial market to reveal less information about the firm's fundamentals. (More generally, the comparative static in Proposition 1 would hold even if agents have ex ante heterogeneous hedging needs, providing that the marginal participating agent adds more trading due to hedging than trading due to information, relative to the average participating agent.)

[15]Manzano and Vives (2011) give a mathematical definition of stability. One way to think about stability is in terms of condition (A-11) in the proof of Proposition 1. The right hand side (RHS) describes agents' demands, which in turn depend on price efficiency (this can be seen explicitly from (A-12)). The left hand side (LHS) of (A-11) describes how prices must behave to clear the market, given agents' demands on the RHS. Equilibrium price efficiency is a fixed point of this relation. Moreover, the RHS is increasing in ρ_k, at least in the neighborhood of any solution. If the RHS crosses the 45^o line from below, the corresponding equilibrium is unstable in the following sense: A small upwards perturbation in agents' beliefs about price efficiency affects agents' demands and increases the RHS. To preserve market clearing, this then pushes ρ_k up, and precisely because the RHS crosses the 45^o line from below, the change in ρ_k is greater than the original perturbation in agents' beliefs about ρ_k, i.e., instability.

3.4 Expected utility from participation

We next turn to agents' participation decisions. To do so, we first characterize an agent's expected utility from participation. As noted in the introduction, a concise representation of expected utility in economies of this type has proved challenging to obtain in related work. The key to a concise representation is Corollary 1's link between the risk premium $\mathbb{E}[X_k - P_k]$, the aggregate amount of risk to share, S_k, and the endogenous covariance between returns $X_k - P_k$ and asset cash flows X_k. Looking ahead, a concise representation is important in order to establish the strategic complementarity of participation decisions (Proposition 3), which in turn allows us to take comparative statics in participation costs.

We define the single-asset analogues of $\mathcal{U}_i^A(e_i)$ and $\mathcal{U}_i^0(e_i)$ by

$$\begin{aligned}\mathcal{U}_{ik}^A(e_{ik}) &\equiv \mathbb{E}\left[\max_{\theta_{ik}} \mathbb{E}\left[u\left(\left(\theta_{ik}+e_{ik}\right)\left(X_k-P_k\right)+e_{ik}P_k\right)|y_i, e_{ik}, P_k\right]|e_{ik}\right], \\ \mathcal{U}_{ik}^0(e_{ik}) &\equiv \mathbb{E}\left[u\left(e_{ik}X_k\right)|e_{ik}\right] = -\exp\left(-\gamma e_{ik}\mathbb{E}[X_k] + \frac{\gamma^2 e_{ik}^2}{2\tau_X}\right), \end{aligned} \tag{25}$$

i.e., respectively, the expected utility from participation in the market for asset k (active trading), and the expected utility from not participating in the market for asset k. As before, we write both quantities conditional on the exposure realization e_{ik}, and write $\mathcal{U}_{ik}^A$ exclusive of the participation cost. Note that non-participation expected utility follows from the usual certainty equivalence formula.

Proposition 2 *In a linear equilibrium with price efficiency ρ_k,*

$$\mathcal{U}_{ik}^A(e_{ik}) = \left(d_{ik}(\rho_k) D_k(\rho_k)\right)^{-1/2} \exp\left(-\frac{1}{2}\Lambda_k(\rho_k)\left(e_{ik}-S_k\right)^2\right)\mathcal{U}_{ik}^0, \tag{26}$$

where

$$d_{ik}(\rho_k) \equiv \frac{\mathrm{var}\left(X_k|e_i, P_k\right)}{\mathrm{var}(X_k|y_i, e_i, P_k)}, \tag{27}$$

$$D_k(\rho_k) \equiv \frac{\mathrm{var}\left(X_k - P_k|e_i\right)}{\mathrm{var}\left(X_k|e_i, P_k\right)}, \tag{28}$$

$$\Lambda_k(\rho_k) \equiv \frac{\left(\frac{\mathrm{cov}(P_k, e_{ik})}{\mathrm{var}(e_{ik})} + \gamma\mathrm{cov}(X_k - P_k, X_k)\right)^2}{\mathrm{var}(X_k - P_k|e_i)}. \tag{29}$$

Moreover, participation n_k affects $d_{ik}(\rho_k)$, $D_k(\rho_k)$, and $\Lambda_k(\rho_k)$ only via price efficiency ρ_k.

In light of Proposition 2, we often write $\mathcal{U}_{ik}^A(e_{ik};\rho_k)$ to make explicit its dependence on price efficiency ρ_k. A participating agent's gain relative to non-participation utility $\mathcal{U}_{ik}^0$ is represented by the benefits Λ_k and D_k, which stem from risk-sharing and are the same for all agents, no matter how precise or imprecise their private information; and d_{ik}, which stems from the the advantages of more precise private information.

We note that the risk-sharing gains are increasing in the absolute difference of an agent's exposure shock e_{ik} relative to the average endowment in the economy, as such agents have more to gain from trade. These risk-sharing gains are also increasing in Λ_k, defined in (29), and a composite of three component terms. To interpret these terms, note first that Lemma 2 implies $\text{cov}(X_k - P_k, X_k)$ is positive; that Lemma 5 implies $\text{cov}(P_k, e_{ik})$ is negative; and that Lemma 6 implies that the combined numerator term $\frac{\text{cov}(P_k,e_{ik})}{\text{var}(e_{ik})} + \gamma \text{cov}\left(X_k - P_k, X_k\right)$ is positive.[16] The three component terms in Λ_k have the following interpretation. First, agents' final wealth depends on their exposure shocks e_{ik} only to the extent that they cannot hedge these at the trading stage: as standard in models with CARA preferences, agents undo their risk exposure by trading against it (see (15)). Thus, they prefer prices to covary as little as possible with their exposures, i.e., for $|\text{cov}(e_{ik}, P_k)|$ to be small. Second, welfare is decreasing in $\text{cov}(P_k, X_k)$, which is closely related to price efficiency ρ_k (in particular, by Lemma 2 it is increasing in price efficiency), capturing the Hirshleifer effect that risk-sharing is hampered when agents have accurate information at the time of trading. Third, welfare is decreasing in the variance of trading profits, $\text{var}(X_k - P_k|e_{ik})$, as one would expect.

Turning to D_k, the denominator $\text{var}\left(X_k|e_{ik}, P_k\right)$ is decreasing price efficiency ρ_k. Loosely speaking, one would also expect the numerator $\text{var}\left(X_k - P_k|e_{ik}\right)$ to decrease in price efficiency, since in this case P_k is more closely related to X_k, and so $X_k - P_k$ is less volatile. The proof of Proposition 3 establishes that the numerator indeed falls, and moreover is the dominant effect, so D_k is decreasing in price efficiency, and hence increasing in participation. The key step in the proof of this fact is to use Lemma 5's result that the demand curve slopes down.

The gains from trading on private information are captured by d_{ik}, which has a form familiar from existing literature. In particular, d_{ik} directly measures the extent to which observing the private signal y_{ik} improves agent i's forecast of the cash flow X_k, relative to forming forecasts based only on the publicly observable price P_k and the agent's private exposure e_{ik}.

An immediate and intuitive implication of Proposition 2 is that expected utility is increasing in the precision of an agent's information, τ_i. So consistent with our initial conjecture, linear equilibria are characterized by some n_k such that agents $i \leq n_k$ participate, and agents $i > n_k$ do not.

[16] This last implication is established in (A-29) in the proof of Proposition 3.

3.5 Strategic complementarity in participation decisions

Next, we show that agents' individual participation decisions exhibit strategic complementarity. As discussed in the introduction, this is the key analytical result in the paper. And as noted earlier, the key step in the proof is Lemma 6's implication that the information in prices affects demand more than the information in exposure shocks.

Proposition 3 *As participation n_k increases, each individual agent's gain from participation $\mathcal{U}^A_{ik}(e_{ik};\rho_k) - \mathcal{U}^0_{ik}(e_{ik})$ increases.*

Economically, the key driving force behind strategic complementarity is that, as participation n_k increases, price efficiency ρ_k drops (Proposition 1). Loosely speaking, lower price efficiency increases the amount of risk-sharing that the financial market enables. Specifically, the risk sharing function of the financial market is to enable agents with high idiosyncratic exposures u_{ik} to share cash flow risk $u_{ik}X_k$ with other agents with low idiosyncratic exposures.

Lower price efficiency corresponds to agents having less information about the cash flow X_k, which makes risk sharing easier to sustain as in Hirshleifer (1971). However, and as discussed in the context of the public information benchmark of subsection 3.1, the risk sharing benefits of less efficient prices must be compared to the potential costs that arise from more volatile prices, since if prices are less efficient, they are relatively more exposed to the aggregate exposure shock Z_k (essentially, the discount rate), and this can easily lead to greater volatility. Proposition 3 establishes that the benefits of lower price efficiency always dominate the potential costs.

4 The effect of declining indexing costs

We are now in a position to address our main question: How does a decline in the cost of indexing, as represented by the parameter κ_1, affect equilibrium outcomes?

For use throughout this section: Since the same number of agents participate in all the non-index synthetic assets $k \neq 1$, by Proposition 1, price efficiency is the same for all such assets. We write ρ_{-1} for this common level of price efficiency, along with ρ_1 for the price efficiency of the index asset.

We start by explicitly writing the expected utilities for agents who fully participate, who index, and who do not participate. Using the asset-by-asset utility for agents who do not participate in a given asset market, the expected utility of agents who do not participate is:

$$\mathcal{U}_i^0(e_i) = -\left|\prod_{k=1}^{m} \mathcal{U}_{ik}^0(e_{ik})\right|. \tag{30}$$

Similarly, the expected utility of agents who fully participate in financial markets is:

$$\mathcal{U}_i^A(e_i; \rho_1, \rho_{-1}) = -\left|\prod_{k=1}^{m} \mathcal{U}_{ik}^A(e_{ik}; \rho_k)\right|. \tag{31}$$

The expected utility of indexers is a mixture of these two cases:

$$\mathcal{U}_i^I(e_i; \rho_1) = -\left|\mathcal{U}_{i1}^A(e_{i1}; \rho_1) \prod_{k=2}^{m} \mathcal{U}_{ik}^0(e_{ik})\right|. \tag{32}$$

Relative to "active traders," these agents miss out on the gains from trading assets outside the index ($k = l+1, \ldots, m$), as well from trading assets covered by the index in different proportions to their index weights (as represented by non-zero positions in assets $k = 2, \ldots, l$). On the other hand, indexing agents benefit from lower participation costs, $\kappa_1 < \kappa$. (Recall that $\mathcal{U}_i^A$ and $\mathcal{U}_i^I$ are defined as exclusive of participation costs κ and κ_1.)

From Proposition 2, an agent's welfare $\mathcal{U}_{ik}^A(e_{ik}; \rho_k)$ associated with participation in the market for asset k is increasing in the precision τ_i of the agent's private information, and hence decreasing in i. Consequently, the sets of agents who participate fully, N, and who participate either fully or via indexing, $N \cup N_1$, must both consist of all agents with precision levels below some cutoff. That is, both N and $N \cup N_1$ are lower intervals of $[0, 1]$. Accordingly, define

$$\begin{aligned} n_1 &\equiv \sup N \cup N_1, \\ n_{-1} &\equiv \sup N. \end{aligned}$$

Hence n_1 is the number of agents who trade the index asset 1, while n_{-1} is the number of agents trading all the remaining assets. Note that certainly $n_{-1} \leq n_1$, i.e., more agents trade the index asset than any other asset, because all agents participate either fully or via indexing trade the index asset. The number of agents who index, in the sense of participating *only* via indexing, is $n_1 - n_{-1}$.

In particular, there are two distinct possible types of equilibrium. An *indexing equilibrium* is one in which $n_1 - n_{-1} > 0$; and a *no-indexing equilibrium* is one in which $n_1 - n_{-1} = 0$.

Given these observations, an equilibrium is fully characterized by the values of n_1 and n_{-1}, i.e., by the marginal agents who trade the index asset 1 and other assets $k \neq 1$. Accordingly, we frequently denote a specific equilibrium by (n_1, n_{-1}).

Note that because of the strategic complementarities established in Proposition 3, participation is self-reinforcing, and so there may simultaneously exist equilibria with high participation levels, and equilibria with low participation levels. Whether such multiplicity in fact arises depends on the distribution of information precisions, given by $\{\tau_i\}$, on which we have imposed almost no assumptions. We state our results below so as to allow for equilibrium multiplicity.

4.1 The prevalence of indexing

With these preliminaries in place, we can state our main result on how indexing costs affect participation decisions. In particular, we consider what happens as the indexing cost κ_1 falls. This corresponds to falling fees, greater availability, and greater awareness of products such as low-cost index funds and ETFs. It may also reflect an increase in public awareness of the standard advice given by finance academics.

We take this comparative static while leaving the cost of full participation, κ, unchanged; however, our results remain qualitatively unchanged if κ also falls, but does by less than κ_1.

As indexing costs κ_1 fall, indexing equilibria are easier to support, and feature more agents indexing and fewer agents fully participating. Conversely, no-indexing equilibria are harder to support.

Proposition 4
(A) For any indexing cost κ_1, at least one equilibrium exists.
(B) As the indexing cost falls, indexing equilibria are easier to support, and feature more indexing. That is, for indexing costs κ_1, κ_1' such that $\kappa_1 < \kappa_1'$:
(i) If an indexing equilibrium exists at κ_1', an indexing equilibrium exists at κ_1 also. Moreover, indexing equilibria at κ_1 feature more total participation, i.e., higher values of n_1.[17]
(ii) If an indexing equilibrium exists at κ_1', the maximum amount of indexing in equilibria at κ_1 exceeds the maximum amount of indexing in equilibria at κ_1'.
(C) As the indexing cost falls, no-indexing equilibria are harder to support. That is, for indexing costs κ_1, κ_1' such that $\kappa_1 < \kappa_1'$, if a no-indexing equilibrium exists at κ_1, a no-indexing equilibrium exists at κ_1' also.

[17]If multiple indexing equilibria exist, this statement should be interpreted as in Milgrom and Roberts (1994) as referring to *extremal* equilibria, i.e., equilibria with minimum and maximum participation levels. Note also that Milgrom and Roberts (1994)'s analysis ensures that extremal equilibria are well defined (e.g., an equilibrium with maximum participation indeed exists).

The economics behind Proposition 4 are relatively straightforward given the strategic complementarity of participation decisions (Proposition 3). As indexing costs κ_1 fall, this directly increases the gain to participation-via-indexing. This in turn raises the gain to other agents of participation-via-indexing, amplifying the original effect. At the same time, the fall in indexing costs κ_1 *raises* the marginal cost $\kappa - \kappa_1$ of full participation, with analogous effects: there is both a direct fall in full participation, which reduces the gain to other agents of fully participating, in turn further reducing full participation. Combining, the amount of indexing increases, with entry at both margins—some people who did not previously participate start trading the index, and some people who previously traded non-index assets switch to trading just the index.

Given the strategic complementarity of participation decisions, the proof of Proposition 4 largely consists of applying monotone comparative statics (Milgrom and Roberts (1994)). The main difficulties, which are handled in the formal proof, lie in simultaneously allowing for the possibility of indexing and no-indexing equilibria, and in the fact that a fall in κ_1 simultaneously makes index participation more attractive but full participation less attractive.

4.2 Indexing and price efficiency

As discussed, reductions in indexing costs affect agents' trading decisions both directly, and indirectly because of changes in price efficiency. Indeed, by Proposition 2 the spillover effect of other agents' trading decision can be summarized entirely by price efficiency.

Here, we collect our analysis's implications for how reductions in indexing costs affect price efficiency. A necessary preliminary is to relate the price efficiency of synthetic assets, which is what our analysis makes direct predictions about, to the price efficiency of actual assets. To do so, it is in turn useful to define the *relative price efficiency* of assets j, k by

$$\operatorname{var}(\tilde{X}_j - \tilde{X}_k | \tilde{P}_j - \tilde{P}_k)^{-1}.$$

That is, the relative price efficiency of assets j and k is the extent to which the relative price of assets j and k forecasts the relative future cash flows of this same pair of assets. Empirical papers such as Bai, Philippon, and Savov (2016) estimate relative price efficiency because they include time fixed effects.

Lemma 7 *The relative price efficiency of any pair of assets in the index, and also of any pair of assets outside the index, is measured by ρ_{-1}.*

Index price efficiency, i.e., $\operatorname{var}(X_1|P_1)^{-1}$, is directly measured by ρ_1 (recall (17)). For a broad-based index, such as the S&P 500, index price-efficiency is close to market price efficiency.

The following is then immediate from Propositions 1 and 4:

Corollary 2 *Let κ_1 be an indexing cost such that an indexing equilibrium exists. If the indexing cost falls, then index price efficiency falls, while relative price efficiency of assets both inside and outside the index rises.*[18]

We reiterate that agents benefit from the fall in index price efficiency, because it increases the gains from trade to agents trading the index.

A separate and basic prediction of our analysis is that the index asset has lower price efficiency than all other synthetic assets, i.e., $\rho_1 < \rho_{-1}$. This prediction maps to a statement about actual assets. Assets covered by the index are linear combinations of the index asset with other synthetic assets. In contrast, assets outside the index coincide with synthetic assets. Intuitively, an "averaging" argument then suggests that assets covered by the index have lower price efficiency. The following result makes this intuition precise.

Lemma 8 *In any indexing equilibrium, the price efficiency of assets covered by the index is strictly lower than the price efficiency of assets not covered by the index, i.e., for j, k such that $j \leq l < k$, $\operatorname{var}\left(\tilde{X}_j|\tilde{P}_j\right)^{-1} < \operatorname{var}\left(\tilde{X}_k|\tilde{P}_k\right)^{-1}$.*

The proof of Lemma 8 consists of establishing that while the equilibrium price of assets $k > l$ outside the index take the form

$$\tilde{P}_k = \text{constant} + b_m \rho_{-1} \tilde{X}_k - b_m \tilde{Z}_k,$$

for some constant b_m, the equilibrium price of assets $j \leq l$ covered by the index take the form

$$\tilde{P}_j = \text{constant} + b_l \tilde{\rho} \tilde{X}_j - b_l \tilde{Z}_j + X_0 + Z_0, \tag{33}$$

for some constant $\tilde{b}_l$, $\tilde{\rho} < \rho_{-1}$, and random variables X_0 and Z_0 that are independent of both $\tilde{X}_j$ and $\tilde{Z}_j$. Moreover, the random variables X_0 and Z_0 are linear combinations of, respectively, cash flows of stocks in the index, $\left\{\tilde{X}_{k'}\right\}_{k'=1}^{l}$, and exposure shocks related to stocks in the index, $\left\{\tilde{Z}_{k'}\right\}_{k'=1}^{l}$. So in particular, X_0 and Z_0 correspond to correlated factors that index trading introduces to stocks covered by the index.

[18]If multiple equilibria exist, Corollary 2, and also Corollary 3 below, should be understood as in Proposition 4. Specifically, for Corollary 2: Across indexing equilibria, index price efficiency falls in the extremal indexing equilibria; and the maximum level of relative price efficiency in any indexing equilibrium rises.

As discussed in the introduction, one might have conjectured that the fall in the price efficiency of the index associated with lower indexing costs would make indexing *less* attractive for uninformed investors, thereby generating a countervailing force. Instead, our analysis shows that the entry of uninformed investors is self-reinforcing: as more such investors enter, and price efficiency drops, indexing becomes more attractive rather than less, attracting still more uninformed investors.

Conversely, one might have conjectured that the rise in relative price efficiency of assets covered by the index would make individual trades more attractive. Again, this is not the case: this increase makes trading individual assets less attractive for uninformed investors, and so is again self-reinforcing.

4.3 Indexing and expected utilities

As the cost κ_1 of indexing falls, the expected utility of indexing agents increases, reflecting both the lower cost, and also the lower price efficiency of the index asset. In contrast, the expected utility of fully participating agents is subject to conflicting forces. On the one hand, fully participating agents benefit from the lower price efficiency of the index asset. On the other hand, they are harmed by the higher price efficiency of other assets.

Combining:

Corollary 3 *Let κ_1 be an indexing cost such that an indexing equilibrium exists. If the indexing cost falls, the expected utility of indexing agents increases. For agents who fully participate, the share of trading gains stemming from trading the index asset increase.*

4.4 Indexing, reversals, and informed trading

A direct implication of market-clearing (7) and agents' trading decisions (15) is that

$$\frac{1}{\gamma}\mathbb{E}\left[X_1 - P_1|P_1\right]\frac{1}{n_1}\int_0^{n_1}\frac{1}{\operatorname{var}\left(X_1|y_{i1},e_{i1},P_1\right)} = S_1 + \mathbb{E}\left[Z_1|P_1\right], \tag{34}$$

with analogous identities for other assets. Moreover, from Lemma 5, we know $\mathbb{E}\left[Z_1|P_1\right]$ is decreasing in P_1, i.e., prices and exposure shocks are negatively correlated. Consequently, our framework naturally generates a reversal pattern in prices, with high prices today associated with lower expected returns.[19]

[19] Consequently, if an investor observes a low price for an asset, and has no exposure to economic shocks, he should take a long position in the asset, since its conditional expected return is high. That is, an investor can profit from buying "value" stocks. Although this point is often overlooked, it is nonetheless a standard implication of models of the type we consider here (see, e.g., Biais, Bossaerts, and Spatt, 2010).

The strength of reversals is captured by the steepness of the negative slope of $\mathbb{E}[X_1 - P_1|P_1]$, i.e., when this relation is strongly negative, the expected returns following high prices are much lower than following low prices. This is determined by price efficiency, and hence in turn by participation decisions:

Lemma 9 $\frac{\partial}{\partial P_1} E[X_1 - P_1|P_1]$ *is negative, and decreases (i.e., becomes further from* 0*) as price efficiency* ρ_1 *declines, and hence as the cost of index participation* κ_1 *decreases.*

Economically, high prices are more likely when the average exposure Z_1 is high. Consequently, agents who observe a high asset price are unable to fully infer whether the high price indicates a high future cash flow, or a high value of Z_1 (i.e., high aggregate unwillingness to buy the asset). Although all agents face this inference problem, agents with more precise private signals are better able to resolve it, and to shy away from the asset when future cash flows are in fact low. To express this formally, fix an arbitrary $\hat{n} \in (0, n_1)$, so that agents $[0, \hat{n}]$ correspond to relatively well-informed investors, while agents $[\hat{n}, n_1]$ correspond to relatively uninformed investors. From agents' trades (15), the difference in the average position of well-informed to uninformed investors is given by

$$\frac{1}{\gamma}\left(\frac{1}{\hat{n}}\int_0^{\hat{n}} \tau_i di - \frac{1}{n_1 - \hat{n}}\int_{\hat{n}}^{n_1} \tau_i di\right)(X_1 - P_1). \tag{35}$$

That is, informed investors own a disproportionately high share of the asset precisely when returns are high, and a disproportionately low share precisely when returns are low.

Conversely, relatively uninformed investors own a disproportionately high share of the asset precisely when returns are low. By Lemma 9, it further follows that relatively uninformed investors own a high share when current prices are high. Hence relatively uninformed investors engage in behavior that resembles "trend chasing," and experience lower average returns. (We stress that all agents in our model are fully rational.)

5 Empirical implications

The sharpest predictions of our model concern price efficiency (subsection 4.2). To recap, our analysis predicts that (i) as indexing becomes easier, relative price efficiency rises, (ii) price efficiency is lower for stocks covered by the index than for those outside it, and (iii) as indexing becomes easier, the price efficiency of the index as a whole decreases.

A number of recent empirical papers have studying related predictions, especially in regard to ETFs. With regard to (i), Bai, Philippon, and Savov (2016) and Farboodi, Matray, and Veldkamp (2018) find that relative price efficiency has trended upwards over approximately the last 50 years, over broadly the same time period in which indexing has become more prevalent. Over a more recent period, Glosten, Nallareddy, and Zou (2016) document that relative price efficiency increases precisely as ETF ownership of the underlying shares increases.[20]

With regard to (ii), Farboodi, Matray, and Veldkamp (2018) show that for stocks that have been included in the S&P 500 at some point in time, price efficiency is lower during periods in which they are included. Ben-David, Franzoni, and Moussawi (2018) document that ETF ownership increases stock volatility, consistent with the hypothesis that ETFs enable "liquidity shocks [to] propagate." Also consistent with our analysis, they provide evidence that the increase in volatility is *not* due to increased price efficiency. Using a measure of weak-form price efficiency, Coles, Heath, and Ringgenberg (2017) show that inclusion in the Russell 2000 reduces price efficiency. Antoniou, Subrahmanyam, and Tosun (2019) show that ETF ownership is associated with a weakening of the link between a firm's investment and its own stock price, suggesting that managers believe that ETF ownership reduces price efficiency.[21] Similarly, Brogaard, Ringgenberg, and Sovich (forthcoming) show that firms that use commodities covered by leading commodity indices make less efficient production decisions than firms that use commodities outside these indices.

While prediction (iii) is harder to directly test, a closely related prediction is that, as indexing increases, informed trading profits will stem increasingly from "timing" strategies based on the entire index, rather than individual asset trades. (See Corollary 3, with the caveat that expected utilities are distinct objects from expected profits.) There is at least some empirical evidence supporting this prediction. AQR document that the correlation between hedge fund returns and market returns has risen from 0.6 to 0.9 over the last two decades.[22] Related, Stambaugh (2014) documents a decline in asset-selection strategies by active mutual funds over the same period. Also related, and using data since 2000, Gerakos, Linnainmaa, and Morse (2017) show that a significant fraction of returns generated by active

[20] We also note that Farboodi, Matray, and Veldkamp (2018) additionally show that relative price efficiency has declined for stocks outside the S&P 500, a finding that is inconsistent with our model. Among other things, these authors emphasize the importance of accounting for changes in firm size, which is outside the scope of our analysis. Israeli, Lee, and Sridharan (2017) estimate similar empirical specifications to Glosten, Nallareddy, and Zou (2016), but use lagged changes in ETF ownership, and show that these are associated with decreases rather than increases in relative price efficiency.

[21] We also note that Li, Liu, and Sun (2019) document just the opposite relation, i.e., that ETF ownership is associated with a strengthening of the link between a firm's investment and its own stock price These authors present additional evidence that the effect is due to ETF ownership leading a firm's stock price to load on index fundamentals, which is consistent with our analysis (specifically, this corresponds to the term X_0 in (33)).

[22] See "Hedge fund correlation risk alarms investors," Financial Times, June 29th, 2014.

mutual funds stem from market timing strategies.

Our model can also be used to study the effect of index inclusion on return variances and covariances. For example, and as one would expect, index inclusion introduces a common component to stock prices; see discussion immediately following Lemma 9. Da and Shive (2018) and Leippold, Su, and Ziegler (2016) present evidence that individual stock correlations have risen with ETF ownership. As noted, Ben-David, Franzoni, and Moussawi (2018) present evidence that ETF ownership raises the volatility of individual stock prices.

The analysis in subsection 4.4 predicts that relatively uninformed ownership of an asset increases when prices are high, and that this is followed by low subsequent returns. This is consistent with the empirical evidence in Ben-Rephael, Kandel, and Wohl (2012) for the market as a whole (i.e., the index asset), and in Jiang, Verbeek, and Wan (2017) and Grinblatt et al. (2016) in the cross-section.[23] Furthermore, Lemma 9 is consistent with the empirical findings of Baltussen, van Bekkum, and Da (2019), who use international data to document that negative serial correlation in index returns is associated with greater indexing.

Finally, it is worth noting that our model does *not* generate a price premium for index inclusion, contrary to at least some empirical evidence. The reason is that changes in participation in our model are always accompanied by changes in the effective supply of the asset being traded, by virtue of assumption (2).[24] Because of this, index inclusion affects prices only via its effect on price efficiency; and since price efficiency falls, this leads to a fall in prices. In contrast, if one were to relax assumption (2), the effect on prices would be determined by the relative strength of changes in price efficiency and changes in the aggregate risk-sharing capacity of participating agents. In general, we emphasize again that our model's primary empirical predictions operate via price efficiency.

6 Discussion

We have analyzed what we believe is the most direct impact of a decline in the costs of indexing, namely those that stem from the entry of new participants into financial markets in response to lower costs, along with substitution of other traders away from full participation to index-only strategies. Nonetheless, our analysis inevitably omits other potentially

[23]Less directly, this prediction is also consistent with the finding in Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014) that the fraction of mutual fund returns stemming from timing strategies is greater in recessions. Specifically, in our setting informed investors should shift into the index when index prices are low, and out of the index when index prices are high. To the extent to which the second half of this strategy is constrained by difficulties shorting the index, this generates the findings of Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014). (We should also note that the same authors suggest a distinct explanation in Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016).)

[24]Recall, in turn, that we use this assumption in the proof of Proposition 2.

important forces.

6.1 Multiple indices

We have considered the case in which a single index trades, and in empirical discussions often think of this single index as a broad-based market index such as the S&P500. In reality of course, a large number of indices co-exist, and investment vehicles such as ETFs are increasingly available on this assortment of indices (see, for example, Lettau and Madhaven (2018) and Easley et al. (2018)).

Our analysis straightforwardly extends to multiple indices in the case that the indices are statistically independent. To the extent to which indices often relate to asset pricing factors, and these factors have been constructed to have minimal correlation with each other, this case is empirically relevant.

The analysis of multiple *correlated* indices would require a larger departure from our current analysis, which is based on a set of synthetic assets that are constructed to be statistically independent. We leave the exploration of this case for future research.

6.2 Intensive margin of information acquisition

While the participation decision in our model can be thought of as an information acquisition decision on the extensive margin, we have abstracted away from the intensive margin of agents' information acquisition efforts. In an earlier draft of this paper we fully analyzed a model that includes this force. In brief, consider the consequences of an exogenous increase in indexing at the expense of active trading, with the increase concentrated among agents with (relatively) low precision private signals. The direct effect is an increase in relative price efficiency of individual assets, with no effect on index price efficiency (because, by assumption, the increase in indexing consists of agents not trading individual assets, with no increase in trade of the index asset itself). Allowing agents to reoptimize the precision of their private signals, these changes in price efficiency in turn induce agents to acquire less precise signals about non-index assets, since this information is now less valuable; and in turn to substitute their information collection activities towards acquiring information about economic aggregates that are important for the index. The net effect of this change in the intensive margin is that price efficiency increases for both the index and non-index assets. The same economic forces as in our current analysis then lead to a reduction in expected utility, since (exactly as in Proposition 2) agents prefer to participate in financial markets where price efficiency is low.

So to summarize: by themselves, information acquisition decisions in response to an exogenous rise in indexing end up reducing rather than increasing the welfare of agents with low-precision private signals, who can be interpreted as retail investors.

More generally, our analysis highlights that the welfare consequences of shifts in indexing—or, indeed, or other changes to financial markets—depend critically on how such shifts affect price efficiency. At least when the gains from trade that underpin financial markets are driven by the benefits from risk sharing, as is the case of many standard models, agents generally prefer low levels of price efficiency.

6.3 Indexing and firm performance

Increases in index investing have led to a variety of concerns about the effect of indexing on firm performance, including (a) the fear that indexing investors will spend less effort on firm governance, and (b) the fear that indexing results in extensive common ownership, leading to reduced competition between firms, thereby reducing consumer surplus.[25] We have deliberately focused our analysis on indexing's effects in an endowment economy, in order to clearly delineate what we believe is an important channel.

The most direct way to extend our analysis beyond an endowment economy would be to consider how changes in price efficiency affect firms' cash flows (see Bond, Goldstein, and Edmans (2012) for a survey of the literature). Such an extension would be analytically feasible; papers such as Sockin and Xiong (2015), Bond and Goldstein (2015), Sockin (2018), and Goldstein and Yang (2018) all contain competitive models of financial markets with asymmetric information in which economic agents extract information from financial prices and use this information to affect firm cash flows.

7 Conclusion

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting (Grossman and Stiglitz (1980); Hellwig (1980); Diamond and Verrecchia (1981)). Individuals must incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing strategies. Individuals' participation decisions exhibit strategic complementarity, and consequently, equilibrium effects reinforce the direct consequences of declining costs of indexing. As indexing becomes cheaper (1) indexing increases, while individual stock trading decreases; (2) aggregate price

[25] For a discussion of the first point, along with some evidence against, see for example Appel, Gormley, and Keim (2016). For a discussion of the second point, see, for example, Azar, Schmalz, and Tecu (2018) and Schmalz (2018).

efficiency falls, while relative price efficiency increases; (3) the welfare of relatively uninformed traders increases; (4) for well-informed traders, the share of trading gains stemming from market timing increases, and the share of gains from stock selection decreases; (5) market-wide reversals become more pronounced. We discuss empirical evidence for these predictions.

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Appendix A

Note: Throughout the appendix, we frequently omit asset subscripts in order to enhance notational transparency.

Results omitted from main text

Lemma A-1 *Suppose X is a normally distributed random variable, and that an information set $\mathcal{F}$ consists of a set of normally distributed random variables. Then the derivative of the conditional expectation $\mathbb{E}[X|\mathcal{F}]$ with respect to a realization $\hat{X}$ of X is*

$$\frac{\partial}{\partial \hat{X}} \mathbb{E}[X|\mathcal{F}] = 1 - \frac{\operatorname{var}(X|\mathcal{F})}{\operatorname{var}(X)}.$$

Proof of Lemma A-1 Let Σ_{22} be the variance matrix of the random variables in $\mathcal{F}$; and Σ_{12} be the row vector of covariances between X and the random variables in $\mathcal{F}$. The vector of coefficients from the regressing each variable in $\mathcal{F}$ on X is $\frac{\Sigma_{12}}{\operatorname{var}(X)}$. So by the properties of multivariate normality,

$$\frac{\partial}{\partial \hat{X}} \mathbb{E}[X|\mathcal{F}] = \Sigma_{12} \Sigma_{22}^{-1} \frac{\Sigma_{12}^{\top}}{\operatorname{var}(X)}.$$

Also from multivariate normality,

$$\operatorname{var}(X|\mathcal{F}) = \operatorname{var}(X) - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^{\top}.$$

Combining these two equations completes the proof.

Lemma A-2 *Let $\xi \in \mathbb{R}^n$ be a normally distributed random vector with mean μ and variance-covariance matrix Σ. Let $b \in \mathbb{R}^n$ be a given vector, and $A \in \mathbb{R}^{n \times n}$ a symmetric matrix. If $I - 2\Sigma A$ is positive definite, then $\mathbb{E}\left[\exp(b^{\top}\xi + \xi^{\top} A \xi)\right]$ is well defined, and given by:*

$$\mathbb{E}\left[\exp\left(b^{\top}\xi + \xi^{\top} A \xi\right)\right] = |I - 2\Sigma A|^{-1/2} \exp\left(b^{\top}\mu + \mu^{\top} A \mu + \frac{1}{2}(b + 2A\mu)^{\top}(I - 2\Sigma A)^{-1}\Sigma(b + 2A\mu)\right). \tag{A-1}$$

Proof of Lemma A-2: Standard result.

Proofs of results stated in main text

Proof of Lemma 1: We focus on cases $l = m$, since the generalization to $l < m$ is trivial. The proof is inductive: Given the existence of an $m \times m$ matrix A with the stated properties, we construct a $2m \times 2m$ matrix B with the same properties. Specifically, define

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ A & -A \end{pmatrix}.$$

With the exception of the symmetry and inversion properties, it is straightforward to see that B has the desired properties. To establish that B is involutory, simply note that

$$BB = \frac{1}{2} \begin{pmatrix} A & A \\ A & -A \end{pmatrix} \begin{pmatrix} A & A \\ A & -A \end{pmatrix} = \frac{1}{2} \begin{pmatrix} AA + AA & AA - AA \\ AA - AA & AA + AA \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2I_m & 0_m \\ 0_m & 2I_m \end{pmatrix} = I_{2m},$$

where I_m denotes the $m \times m$ identity matrix and 0_m denotes the $m \times m$ matrix in which all entries are zero. To establish that B is symmetric, simply note that

$$B^\top = \frac{1}{\sqrt{2}} \begin{pmatrix} A^\top & A^\top \\ A^\top & -A^\top \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ A & -A \end{pmatrix} = B.$$

Finally, for the base case of $m = 1$, simply define $A = (1)$. This completes the proof.

Proof of Lemma 2: For use throughout, note that the price P is normally distributed in a linear equilibrium. Differentiation of market clearing (7) with respect to X gives

$$\frac{\partial}{\partial X} \int_0^n \theta_i di = 0.$$

Substituting in the portfolio θ_i from (15); recalling the property of multivariate normality that conditional variances do not depend on the realizations of random variables; and noting that $\frac{\partial P}{\partial X} = \frac{\text{cov}(P,X)}{\text{var}(X)}$, it follows that

$$\int_0^n \frac{\frac{\partial}{\partial X} \mathbb{E}\left[X | y_i, e_i, P\right]}{\text{var}\left(X | y_i, e_i, P\right)} di = \int_0^n \frac{\frac{\text{cov}(P,X)}{\text{var}(X)}}{\text{var}\left(X | y_i, e_i, P\right)} di. \tag{A-2}$$

By Lemma A-1,

$$\frac{\partial}{\partial X} \mathbb{E}\left[X | y_i, e_i, P\right] = 1 - \frac{\text{var}\left(X | y_i, e_i, P\right)}{\text{var}\left(X\right)}. \tag{A-3}$$

(Note that the RHS of (A-3) is simply the R^2 of regressing cash flows on (y_i, e_i, P).) Substitution of (A-3) into (A-2) yields

$$\left(1 - \frac{\text{cov}\left(P, X\right)}{\text{var}\left(X\right)}\right) \int_0^n \frac{1}{\text{var}\left(X | y_i, e_i, P\right)} di = \int_0^n \frac{1}{\text{var}\left(X\right)} di,$$

which is equivalent to (16), completing the proof of Lemma 2.

Proof of Lemma 3: Differentiation of market clearing (7) with respect to Z yields

$$\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di + \frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di + \int_0^n \frac{\partial \theta_i}{\partial e_i}di + \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i}di = 0. \tag{A-4}$$

Suppose that, contrary to the claimed result, $\frac{\partial P}{\partial Z} = 0$. Then Z and e_i provide no information about the cash flow X, so that $\frac{\partial \theta_i}{\partial \hat{e}_i} = 0$ for all agents. In contrast, the non-informational effect of exposure shocks on the portfolio decision is certainly negative (see (15)). But then the LHS of (A-4) is strictly negative, a contradiction.

Proof of Lemma 4: Consider first the case in which $\frac{\partial P}{\partial X} \neq 0$. The information content of (y_i, e_i, P) is the same as the information content of

$$\left(y_i, \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mathbb{E}[X], \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mathbb{E}[X] + \rho^{-1}(e_i - S)\right) = \left(X + \epsilon_i, X - \rho^{-1}Z, X + \rho^{-1}(u_i + s_i - S)\right).$$

Since ϵ_i, Z, and $u_i + s_i - S$ are independent and all have mean 0, the conditional variance expressions (17) and (18) follow by standard normal-normal updating. Using $\frac{\partial P}{\partial X} = -\rho\frac{\partial P}{\partial Z}$, the corresponding conditional expectation $\mathbb{E}[X|y_i, e_i, P]$ is given by

$$\frac{\mathbb{E}[X|y_i, e_i, P]}{\mathrm{var}(X|y_i, e_i, P)} = \tau_X \mathbb{E}[X] + \rho(\tau_Z + \tau_u)\left(\rho\mathbb{E}[X] - \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial Z}}\right) + \rho\tau_u(e_i - S) + \tau_i y_i. \tag{A-5}$$

Finally, if $\frac{\partial P}{\partial X} = 0$ then neither the price nor the exposure shock e_i contains any information about X; and $\rho = 0$; so (17), (18), and (A-5) are all immediate.

The expressions in Lemma 4 are then immediate from the demand equation (15), completing the proof.

Proof of Lemma 5: From Lemma 4, $\frac{\partial \theta_i}{\partial P} < 0$ for all agents.

If $\frac{\partial P}{\partial X} = 0$, then P contains no information about X, so $\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di = 0$, and (23) is then immediate.

If instead $\frac{\partial P}{\partial X} \neq 0$, then $\frac{\partial P}{\partial X}\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di > 0$ by Lemma 4. By (A-10) and Lemma 4,

$$\frac{\partial P}{\partial X}\int_0^n \frac{\partial \theta_i}{\partial P}di + \frac{\partial P}{\partial X}\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di = -\int_0^n \frac{\partial \theta_i}{\partial y_i}di < 0. \tag{A-6}$$

Hence $\frac{\partial P}{\partial X} > 0$, which (again using (A-6)) implies (23).

Note that the above arguments also establish that $\frac{\partial P}{\partial X} \geq 0$.

Lemma 3 establishes that $\frac{\partial P}{\partial Z} \neq 0$. So to establish $\frac{\partial P}{\partial Z} < 0$, suppose to the contrary that $\frac{\partial P}{\partial Z} > 0$. Then $\rho \leq 0$, and Lemma 4 implies $\int_0^n \frac{\partial \theta_i}{\partial e_i}di < 0$ and $\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i}di \leq 0$. Combined with (23), this in turn implies that the LHS of (A-4) is strictly negative. The contradiction completes the proof.

Proof of Lemma 6: At various points in the proof, we make use of $\rho < 0$ and $\frac{\partial P}{\partial Z} < 0$ (by Lemma 5), and $\frac{\partial \theta_i}{\partial \hat{P}} > 0$ (by Lemma 4).

Re-arranging market-clearing (A-4) gives

$$0 = \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di + \int_0^n \frac{\partial \theta_i}{\partial e_i} di + \left(\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \tilde{e}_i} di} + 1 \right) \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di. \tag{A-7}$$

By Lemma 5 and market-clearing (A-4), we know

$$\int_0^n \frac{\partial \theta_i}{\partial e_i} di + \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di < 0. \tag{A-8}$$

From (22) of Lemma 4, $\frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial \hat{P}} + \frac{\partial \theta_i}{\partial \hat{e}_i} < 0$, which together with $\frac{\partial \theta_i}{\partial \hat{e}_i} > 0$ implies

$$\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di} + 1 < 0.$$

So substituting (A-8) into (A-7) gives

$$\begin{aligned} 0 \quad &> \quad \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di + \int_0^n \frac{\partial \theta_i}{\partial e_i} di - \left(\frac{\frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di} + 1 \right) \int_0^n \frac{\partial \theta_i}{\partial e_i} di \\ &= \quad \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial P} di - \frac{\partial P}{\partial Z} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di \frac{\int_0^n \frac{\partial \theta_i}{\partial e_i} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di}. \end{aligned} \tag{A-9}$$

Using $\frac{\partial P}{\partial Z} < 0$ and the signs established in Lemma 4, inequality (A-9) is equivalent to

$$\frac{\int_0^n \frac{\partial \theta_i}{\partial P} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di} > \frac{\int_0^n \frac{\partial \theta_i}{\partial e_i} di}{\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di},$$

which is in turn equivalent to (24), completing the proof.

Proof of Proposition 1: Differentiation of market clearing (7) with respect to X yields

$$\frac{\partial P}{\partial X} \int_0^n \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial X} \int_0^n \frac{\partial \theta_i}{\partial \hat{P}} di + \int_0^n \frac{\partial \theta_i}{\partial y_i} di = 0. \tag{A-10}$$

Combined with (A-4), it follows that

$$-\frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial Z}} = -\frac{\int_0^n \frac{\partial \theta_i}{\partial y_i} di}{\int_0^n \frac{\partial \theta_i}{\partial e_i} di + \int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i} di}. \tag{A-11}$$

Substituting in from Lemma 4,

$$\rho = \frac{\frac{1}{\gamma}\int_0^n \tau_i di}{\int_0^n \left(1 - \frac{\rho}{\gamma}\tau_u\right) di}, \tag{A-12}$$

and so

$$\rho^2\tau_u - \gamma\rho + \frac{1}{n}\int_0^n \tau_i di = 0,$$

leading to $\rho = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{1}{\tau_u}\frac{1}{n}\int_0^n \tau_i di}$. The comparative static of price efficiency is immediate from the fact that $\frac{1}{n}\int_0^n \tau_i di$ is decreasing in n.

Proof of Proposition 2: The final wealth of agent i, given optimal trading (15), is

$$W_i = e_i P + \frac{\mathbb{E}[X - P|y_i, e_i, P]\,(X - P)}{\gamma \text{var}(X|y_i, e_i, P)}.$$

So by the standard expression for the expected utility of an agent with CARA utility facing normally shocks, combined with simple manipulation, agent i's expected utility at the time of trading is

$$\mathbb{E}\left[u\left(W_i\right)|y_i, e_i, P\right] = -\exp\left(-\gamma\left(e_i P + \frac{1}{2}\frac{\mathbb{E}[X - P|y_i, e_i, P]^2}{\gamma \text{var}(X|y_i, e_i, P)}\right)\right). \tag{A-13}$$

To obtain (26), we proceed in two stages. First, we integrate over realizations of the private signal y_i in (A-13). Second, we integrate over realizations of the price P. Note that the first stage is relatively standard, and similar algebraic arguments can be found in the related literature. Readers familiar with these arguments should proceed directly to the second stage.

For the first stage, define $\xi_i = \mathbb{E}\left[X - P|y_i, e_i, P\right]$ and $A_i = -1/\left(2\text{var}(X|y_i, e_i, P)\right)$. Minor algebraic manipulation of Lemma A-2 implies

$$\mathbb{E}\left[\exp\left(\xi_i^2 A_i\right)|e_i, P\right] = \left(1 - 2A_i \text{var}(\xi_i|e_i, P)\right)^{-\frac{1}{2}} \exp\left(\frac{A_i}{1 - 2A_i \text{var}(\xi_i|e_i, P)}\mathbb{E}[\xi_i|e_i, P]^2\right). \tag{A-14}$$

By the law of total variance,

$$\text{var}(X - P|e_i, P) = \text{var}(\mathbb{E}[X - P|y_i, e_i, P]|e_i, P) + \mathbb{E}[\text{var}(X - P|y_i, e_i, P)|e_i, P]$$

which implies

$$\text{var}(\xi_i|e_i, P) = \text{var}(X|e_i, P) - \text{var}(X|y_i, e_i, P),$$

and so

$$1-2A_i\text{var}(\xi_i|e_i,P)=\frac{\text{var}(X|e_i,P)}{\text{var}(X|y_i,e_i,P)}=d_i,$$

where d_i is as defined in (27). Substitution and straightforward manipulation implies that expression (A-14) equals

$$d_i^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{\mathbb{E}[X-P|e_i,P]^2}{\text{var}(X|e_i,P)}\right),$$

and so

$$\mathbb{E}\left[u\left(W_i\right)|e_i,P\right]=-d_i^{-\frac{1}{2}}\exp\left(-\gamma e_i P-\frac{1}{2}\frac{\mathbb{E}[X-P|e_i,P]^2}{\text{var}(X|e_i,P)}\right), \tag{A-15}$$

completing the first stage.

In the second stage, we integrate over realizations of P. Since $P=(P-\mathbb{E}[P|e_i])+\mathbb{E}[X|e_i]-\mathbb{E}[X-P|e_i]$, the expression in the exponent of (A-15) equals

$$-\frac{1}{2}\frac{\mathbb{E}[X-P|e_i,P]^2}{\text{var}(X|e_i,P)}-\gamma e_i(P-\mathbb{E}[P|e_i])-\gamma e_i\mathbb{E}[X|e_i]+\gamma e_i\mathbb{E}[X-P|e_i]. \tag{A-16}$$

Denote the expected return $X-P$ given exposure e_i by α_e, and recall that $\mathbb{E}\left[e_i\right]=S$:

$$\alpha_e\equiv\mathbb{E}[X-P|e_i]=\mathbb{E}[X-P]-\frac{\text{cov}(P,e_i)}{\text{var}\left(e_i\right)}\left(e_i-S\right). \tag{A-17}$$

Hence

$$\mathbb{E}[X-P|e_i,P]=\frac{\text{cov}\left(X-P,P|e_i\right)}{\text{var}\left(P|e_i\right)}\left(P-\mathbb{E}[P|e_i]\right)+\alpha_e. \tag{A-18}$$

By substitution and Lemma A-2, the expectation of (A-15) conditional on e_i is given by

$$\begin{aligned}\mathbb{E}\left[u\left(W_i\right)|e_i\right] &= -d_i^{-\frac{1}{2}}D^{\frac{1}{2}}\exp\left(-\gamma e_i\left(\mathbb{E}\left[X\right]-\alpha_e\right)\right)\\ &\times\ \exp\left(-\frac{1}{2}\frac{\alpha_e^2}{\text{var}(X|e_i,P)}+\frac{1}{2}\left(\frac{\alpha_e\text{cov}(X-P,P|e_i)}{\text{var}(P|e_i)\text{var}(X|e_i,P)}+\gamma e_i\right)^2\frac{\text{var}(P|e_i)}{D}\right)\end{aligned} \tag{A-19}$$

where

$$D=1+\frac{\text{cov}\left(X-P,P|e_i\right)^2}{\text{var}\left(X|e_i,P\right)\text{var}\left(P|e_i\right)}. \tag{A-20}$$

The law of total variance and (A-18) together yield

$$\text{var}\left(\text{E}\left[X-P|e_i,P\right]|e_i\right)=\text{var}\left(X-P|e_i\right)-\text{var}\left(X|e_i,P\right)=\frac{\text{cov}\left(X-P,P|e_i\right)^2}{\text{var}\left(P|e_i\right)}, \tag{A-21}$$

and substitution into (A-20) delivers (28).

For use below, note also that (A-21) implies that

$$
\begin{aligned}
\frac{\operatorname{var}(P|e_i)}{D} = \frac{\operatorname{var}(P|e_i)\operatorname{var}(X|e_i,P)}{D\operatorname{var}(X|e_i,P)} &= \frac{\operatorname{var}(P|e_i)\operatorname{var}(X-P|e_i) - \operatorname{cov}(X-P,P|e_i)^2}{D\operatorname{var}(X|e_i,P)} \\
&= \frac{\operatorname{var}(X|e_i)\operatorname{var}(X-P|e_i) - \operatorname{cov}(X-P,X|e_i)^2}{D\operatorname{var}(X|e_i,P)} \\
&= \operatorname{var}(X|e_i) - \frac{\operatorname{cov}(X-P,X|e_i)^2}{D\operatorname{var}(X|e_i,P)} \qquad \text{(A-22)}
\end{aligned}
$$

where the penultimate equality follows from the fact that for any random variables r_1, r_2,

$$
\operatorname{cov}(r_1-r_2,r_1)^2 - \operatorname{cov}(r_1-r_2,r_2)^2 = \operatorname{var}(r_1-r_2)\left(\operatorname{var}(r_1)-\operatorname{var}(r_2)\right),
$$

and the final equality follows from (28).

By a combination of algebraic manipulation and (A-20), (28), (A-22), expected utility (A-19) equals

$$
\begin{aligned}
& -(d_iD)^{-\frac{1}{2}}\exp\left(-\gamma e_i\mathbb{E}[X] + \frac{1}{2}\frac{\alpha_e^2}{\operatorname{var}(X|P,e_i)}\left(\frac{\operatorname{cov}(X-P,P|e_i)^2}{D\operatorname{var}(P|e_i)\operatorname{var}(X|e_i,P)} - 1\right)\right) \\
\times\ & \exp\left(\gamma e_i\alpha_e\left(\frac{\operatorname{cov}(X-P,P|e_i)}{D\operatorname{var}(X|e_i,P)}+1\right) + \frac{1}{2}\gamma^2 e_i^2\frac{\operatorname{var}(P|e_i)}{D}\right) \\
=\ & -(d_iD)^{-\frac{1}{2}}\exp\left(-\gamma e_i\mathbb{E}[X] - \frac{1}{2}\frac{\alpha_e^2}{D\operatorname{var}(X|e_i,P)}\right) \\
\times\ & \exp\left(\gamma e_i\alpha_e\frac{\operatorname{cov}(X-P,X|e_i)}{D\operatorname{var}(X|e_i,P)} + \frac{1}{2}\gamma^2 e_i^2\left(\operatorname{var}(X|e_i) - \frac{\operatorname{cov}(X-P,X|e_i)^2}{D\operatorname{var}(X|e_i,P)}\right)\right).
\end{aligned}
$$

By further manipulation, and substitution of α_e using (A-17), expected utility (A-19) equals

$$
\begin{aligned}
& -(d_iD)^{-\frac{1}{2}}\exp\left(-\gamma e_i\mathbb{E}[X] + \frac{\gamma^2 {e_i}^2}{2\tau_X} - \frac{1}{2}\frac{\left(\alpha_e - \operatorname{cov}(X-P,X)\gamma e_i\right)^2}{D\operatorname{var}(X|e_i,P)}\right) \\
=\ & -(d_iD)^{-\frac{1}{2}}\exp\left(-\gamma e_i\mathbb{E}[X] + \frac{\gamma^2 {e_i}^2}{2\tau_X}\right) \\
\times\ & \exp\left(-\frac{1}{2}\frac{\left(\mathbb{E}[X-P] - \operatorname{cov}(X-P,X)\gamma S - \operatorname{cov}(X-P,X)\gamma(e_i-S) - \frac{\operatorname{cov}(P,e_i)}{\operatorname{var}(e_i)}(e_i-S)\right)^2}{D\operatorname{var}(X|e_i,P)}\right).
\end{aligned}
$$

Substituting Corollary 1's expression for $\mathbb{E}[X-P]$ into this last expression yields (29).

Finally, the fact that each of d_i, D, and Λ can be written as functions of exogenous parameters and price efficiency ρ is established in the proof of Proposition 3.

Proof of Proposition 3: For use below, recall that $\frac{\partial P}{\partial X}=\frac{\mathrm{cov}(X,P)}{\mathrm{var}(X)}$, $\frac{\partial P}{\partial Z}=\frac{\mathrm{cov}(Z,P)}{\mathrm{var}(Z)}$, and hence

$$\begin{aligned}
\mathrm{var}\left(P|e_i\right) &= \frac{\mathrm{cov}\left(X,P\right)^2}{\mathrm{var}\left(X\right)^2}\mathrm{var}\left(X\right)+\frac{\mathrm{cov}\left(Z,P\right)^2}{\mathrm{var}\left(Z\right)^2}\mathrm{var}\left(Z|e_i\right), && \text{(A-23)}\\
\mathrm{var}\left(X-P|e_i\right) &= \left(\frac{\mathrm{cov}\left(X-P,X\right)}{\mathrm{var}\left(X\right)}\right)^2\mathrm{var}\left(X\right)+\left(\frac{\mathrm{cov}\left(P,Z\right)}{\mathrm{var}\left(Z\right)}\right)^2\mathrm{var}\left(Z|e_i\right). && \text{(A-24)}
\end{aligned}$$

Note also that Lemmas 2 and 4 directly imply the following expression for the non-informational effect of prices on aggregate demand:

$$\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di=-\frac{1}{\gamma}\frac{1}{\mathrm{cov}\left(X-P,X\right)}. \tag{A-25}$$

We use repeatedly Proposition 1's result that price efficiency decreases in participation n.

To establish the result, we show that each of the three terms d_i, D, and Λ are increasing in participation n. We start with the term d_i, which corresponds to an agent's expected trading gains from her private information. Substitution of (18) delivers

$$d_i=\frac{\tau_X+\rho^2(\tau_Z+\tau_u)+\tau_i}{\tau_X+\rho^2(\tau_Z+\tau_u)}.$$

Because price efficiency ρ is decreasing in participation n, the private gains from information, d_i, are increasing in participation. Economically, when prices convey less information about cash flows X, an agent's private information about X is more valuable.

Next, we consider the term Λ. As a first step, (A-25) implies that

$$\begin{aligned}
\frac{\mathrm{cov}(P,e_i)}{\mathrm{var}(e_i)}+\gamma\mathrm{cov}(X-P,X) &= -\gamma\mathrm{cov}\left(X-P,X\right)\frac{\mathrm{var}\left(Z\right)}{\mathrm{var}\left(e_i\right)}\left(-\frac{1}{\gamma\mathrm{cov}\left(X-P,X\right)}\frac{\mathrm{cov}\left(P,e_i\right)}{\mathrm{var}\left(Z\right)}-\frac{\mathrm{var}\left(e_i\right)}{\mathrm{var}\left(Z\right)}\right)\\
&= -\gamma\mathrm{cov}\left(X-P,X\right)\frac{\mathrm{var}\left(Z\right)}{\mathrm{var}\left(e_i\right)}\left(\frac{1}{n}\int_0^n\frac{\partial\theta_i}{\partial P}di\frac{\mathrm{cov}\left(P,Z\right)}{\mathrm{var}\left(Z\right)}-\frac{\mathrm{var}\left(e_i\right)}{\mathrm{var}\left(Z\right)}\right). && \text{(A-26)}
\end{aligned}$$

Substituting (A-24) and (A-26) into (29), and again using (A-25), gives

$$\begin{aligned}
\Lambda &= \frac{\gamma^2 \mathrm{cov}\left(X-P,X\right)^2 \frac{\mathrm{var}(Z)^2}{\mathrm{var}(e_i)^2}\left(\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}-\frac{\mathrm{var}(e_i)}{\mathrm{var}(Z)}\right)^2}{\left(\frac{\mathrm{cov}(X-P,X)}{\mathrm{var}(X)}\right)^2\mathrm{var}\left(X\right)+\left(\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}\right)^2\mathrm{var}\left(Z|e_i\right)} \\
&= \frac{\frac{\mathrm{var}(Z)^2}{\mathrm{var}(e_i)^2}\left(\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}-\frac{\mathrm{var}(e_i)}{\mathrm{var}(Z)}\right)^2}{\frac{1}{\gamma^2}\frac{1}{\mathrm{var}(X)}+\left(-\frac{\mathrm{cov}(P,Z)}{\mathrm{var}(Z)}\frac{1}{\gamma\mathrm{cov}(X-P,X)}\right)^2\mathrm{var}\left(Z|e_i\right)} \\
&= \frac{\frac{\mathrm{var}(Z)^2}{\mathrm{var}(e_i)^2}\left(\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di-\frac{\mathrm{var}(e_i)}{\mathrm{var}(Z)}\right)^2}{\frac{1}{\gamma^2}\frac{1}{\mathrm{var}(X)}+\left(\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di\right)^2\mathrm{var}\left(Z|e_i\right)}. \qquad \text{(A-27)}
\end{aligned}$$

We next show that a key term in the numerator of (A-27) is negative, i.e.,

$$\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di-\frac{\mathrm{var}\left(e_i\right)}{\mathrm{var}\left(Z\right)}<0. \qquad \text{(A-28)}$$

Inequality (A-28) holds because, by (21), it is equivalent

$$\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di+\frac{\frac{1}{n}\int_0^n \frac{\partial P}{\partial Z}\frac{\partial \theta_i}{\partial \hat{P}}di}{\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial \hat{e}_i}di}<0,$$

and since $\frac{\partial \theta_i}{\partial e_i}=-1$, $\frac{\partial \theta_i}{\partial \hat{P}}>0$ and $\frac{\partial P}{\partial Z}<0$, this inequality is in turn equivalent (24), i.e., prices contain more information than exposure shocks.

Note also that, since $\mathrm{cov}(X-P,X)>0$ (by Lemma 2), equation (A-26) implies that (A-28) is equivalent to

$$\frac{\mathrm{cov}(P,e_i)}{\mathrm{var}(e_i)}+\gamma\mathrm{cov}(X-P,X)>0, \qquad \text{(A-29)}$$

a fact we refer to in the main text.

From (A-28) and (A-27), Λ is decreasing in $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di$. Substitution of (22) into the market clearing condition (A-4), along with the basic property that the non-informational effect of exposure shocks on demand is $\frac{\partial \theta_i}{\partial e_i}=-1$, yields

$$\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di=1+\frac{\rho}{\gamma}\tau_Z. \qquad \text{(A-30)}$$

Hence $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di$ is increasing in price efficiency ρ, and hence is decreasing in participation n, implying that Λ is increasing in participation n.

Finally, we consider the term D, which we start by re-expressing. By the law of total variance,

$$\begin{aligned}\operatorname{var}(X|e_i,P)\operatorname{var}(P|e_i) &= \left(\operatorname{var}(X|e_i) - \frac{\operatorname{cov}(X,P|e_i)^2}{\operatorname{var}(P|e_i)}\right)\operatorname{var}(P|e_i) \\ &= \operatorname{var}(X|e_i)\operatorname{var}(P|e_i) - \operatorname{cov}(X,P|e_i)^2 .\end{aligned}$$

Substituting in (A-23) gives

$$\operatorname{var}(X|e_i,P)\operatorname{var}(P|e_i) = \frac{\operatorname{cov}(Z,P)^2}{\operatorname{var}(Z)^2}\operatorname{var}(Z|e_i)\operatorname{var}(X). \tag{A-31}$$

Also by (A-23), and making use of (A-25),

$$\begin{aligned}\operatorname{cov}(X-P,P|e_i) &= \operatorname{cov}(X,P|e_i) - \operatorname{var}(P|e_i) \\ &= \frac{\operatorname{cov}(X,P)}{\operatorname{var}(X)}(\operatorname{var}(X) - \operatorname{cov}(X,P)) - \frac{\operatorname{cov}(Z,P)^2}{\operatorname{var}(Z)^2}\operatorname{var}(Z|e_i) \\ &= \left(\frac{\frac{\operatorname{cov}(X,P)}{\operatorname{var}(X)}}{\frac{\operatorname{cov}(Z,P)}{\operatorname{var}(Z)}} - \frac{\frac{\operatorname{cov}(Z,P)}{\operatorname{var}(Z)}\operatorname{var}(Z|e_i)}{\operatorname{cov}(X-P,X)}\right)\frac{\operatorname{cov}(Z,P)}{\operatorname{var}(Z)}\operatorname{cov}(X-P,X) \\ &= \left(-\rho + \gamma\operatorname{var}(Z|e_i)\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial P}di\right)\frac{\operatorname{cov}(Z,P)}{\operatorname{var}(Z)}\left(-\frac{1}{\gamma\frac{1}{n}\int_0^n \frac{\partial\theta_i}{\partial P}di}\right)\end{aligned} \tag{A-32}$$

Substitution of (A-31) and (A-32) into (A-20) yields:

$$D = 1 + \frac{\left(-\rho + \gamma\operatorname{var}(Z|e_i)\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial P}di\right)^2}{\gamma^2\operatorname{var}(Z|e_i)\operatorname{var}(X)\left(\frac{1}{n}\int_0^n \frac{\partial\theta_i}{\partial P}di\right)^2}. \tag{A-33}$$

By (A-30) and the fact that $\operatorname{var}(Z|e_i) = \frac{1}{\tau_Z+\tau_u}$,

$$-\rho + \gamma\operatorname{var}(Z|e_i)\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial\theta_i}{\partial P}di = (\gamma - \rho\tau_u)\operatorname{var}(Z|e_i), \tag{A-34}$$

and so

$$D = 1 + \frac{(\gamma-\rho\tau_u)^2\operatorname{var}(Z|e_i)}{\gamma^2\operatorname{var}(X)\left(\frac{1}{n}\int_0^n \frac{\partial\theta_i}{\partial P}di\right)^2}. \tag{A-35}$$

Note also that (A-34) implies that

$$\gamma - \rho\tau_u > 0, \tag{A-36}$$

since, using $\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di = -\frac{\rho}{\gamma \text{var}(Z|e_i)}$ (see Lemma 4), the LHS of (A-34) equals

$$-\rho\left(1+\frac{\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial P}di}{\frac{1}{n}\frac{\partial P}{\partial Z}\int_0^n \frac{\partial \theta_i}{\partial \hat{P}}di}\right). \tag{A-37}$$

Expression (A-37) is strictly positive because $\frac{\partial \theta_i}{\partial \hat{P}} > 0$ and demand slopes down (Lemma 5).

Finally, from Lemma 4 and (18),

$$\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di = -\frac{1}{\gamma}\frac{1}{n}\int_0^n \frac{1}{\text{var}\left(X|y_i, e_i, P\right)}di = -\frac{\tau_X + \rho^2(\tau_Z+\tau_u)+\frac{1}{n}\int_0^n \tau_i di}{\gamma}. \tag{A-38}$$

So as participation n increases, $\left|\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di\right|$ declines, both because price efficiency declines; and because the average signal precision of participating agents, $\frac{1}{n}\int_0^n \tau_i di$, declines.

It then follows from (A-35) and (A-36) that D increases as participation n increases.

Finally, substitution of Proposition 1's expression for ρ into (A-38) yields

$$\left|\frac{1}{n}\int_0^n \frac{\partial \theta_i}{\partial P}di\right| = \frac{\tau_X+\rho^2\tau_Z+\gamma\rho}{\gamma},$$

thereby establishing that D can be expressed as a function of ρ only, completing the proof.

Proof of Proposition 4: For use throughout the proof, we write $A \succeq_i^{\rho_1,\rho_{-1}} 0$ if agent i prefers full participation to non-participation when price efficiency is (ρ_1, ρ_{-1}), i.e., if $\mathbb{E}\left[\mathcal{U}_i^A(e_i;\rho_1,\rho_{-1})\right]\exp(\gamma\kappa) \geq \mathbb{E}\left[\mathcal{U}_i^0(e_i)\right]$. We define the relations $I \succeq_i^{\rho_1} 0$ and $A \succeq_i^{\rho_{-1}} I$ etc analogously, where "I" corresponds to participation via indexing. Note that the comparison between index-participation and non-participation depends only on ρ_1, and not on ρ_{-1}; while the comparison between full participation and index-participation depends only on ρ_{-1}, and not on ρ_1.

Also for use below, define

$$f\left(n\right) \equiv \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{1}{\tau_u}\frac{1}{n}\int_0^n \tau_i di},$$

i.e., price efficiency associated with participation n (see Proposition 1).

Observe that n_1 is an equilibrium level of participation in the index asset 1 only if it is a fixed point of either the function

$$g_A\left(n\right) \equiv \max\left\{i : A \succeq_i^{f(n),f(n)} 0\right\},$$

where both here and below we adopt the adopt the convention that the maximum of an empty set is 0; or of the function

$$g_I(n;\kappa_1) \equiv \max\left\{i : I \succeq_i^{f(n)} 0 \text{ given } \kappa_1\right\}.$$

Moreover, if n_1 is fixed point of g_A, then $(n_1, n_{-1} = n_1)$ is a no-indexing equilibrium if and only if

$$A \succeq_{n_1}^{f(n_1)} I; \tag{A-39}$$

and if n_1 is a fixed point of g_I, then there is an indexing equilibrium with participation n_1 in the index asset if and only if for some $j \le n_1$,

$$I \succ_j^{f(j)} A. \tag{A-40}$$

Finally, define

$$g_{AI}(n;\kappa_1) = \max\left\{i : A \succeq_i^{f(n)} I \text{ given } \kappa_1\right\}.$$

By Propositions 1 and 3, the function g_A (respectively, g_I, g_{AI}) is continuous and weakly increasing in n, and is strictly increasing in the neighborhood of any n for which $g_A(n) \in (0,1)$ (respectively, $g_I(n) \in (0,1)$, $g_{AI}(n) \in (0,1)$).

(A) Let n_1 be a fixed point of $\max\{g_A, g_I\}$ (at least one fixed point exists by Tarski), i.e., $n_1 = \{g_A(n_1), g_I(n_1)\}$. To establish the result, we show there is an $n_{-1} \in [0, n_1]$ such that (n_1, n_{-1}) is an equilibrium. First, consider the case in which $g_A(n_1) < n_1$. So $0 \succ_{n_1}^{f(n_1),f(n_1)} A$. Moreover, $I \approx_{n_1}^{f(n_1)} 0$. Hence $I \succ_{n_1}^{f(n_1)} A$, implying $g_{AI}(n_1) < n_1$. So g_{AI} maps $[0, n_1]$ into itself, and hence (by Tarski) has at least one fixed point in $[0, n_1]$, say n_{-1}. By construction, (n_1, n_{-1}) is an equilibrium. Second, consider the case in which $g_I(n_1) \le n_1$. So $0 \succeq_{n_1}^{f(n_1)} I$. Moreover, $A \approx_{n_1}^{f(n_1),f(n_1)} 0$. Hence $A \succeq_{n_1}^{f(n_1)} I$, establishing that (n_1, n_1) is an equilibrium.

(B) As κ_1 falls, the function g_I strictly increases at any value of n for which $g_I(n) \in (0,1)$. So by Corollary 1 of Milgrom and Roberts (1994), the extremal fixed points of g_I increase. Moreover, condition (A-40) is easier to satisfy. This establishes (i).

For (ii), let $\left(n_1', n_{-1}'\right)$ be the equilibrium with the most indexing at κ_1'. Note that n_{-1}' is the smallest fixed point of $g_{AI}(\cdot;\kappa_1')$. As κ_1 falls, the function g_{AI} decreases. So again by Milgrom and Roberts's Corollary 1, the smallest fixed point of $g_{AI}(\cdot;\kappa_1)$ is weakly smaller than n_{-1}' (and strictly so if $n_{-1}' > 0$). Combined with (i), the result follows.

(C) The function g_A has the same set of fixed points at κ_1 and κ_1'. Condition (A-39) is more demanding to satisfy at κ_1. This completes the proof.

Proof of Lemma 7: Throughout the proof, we use the fact that in equilibrium there exist scalars a and b such that the price of any non-index synthetic asset $k \neq 1$ is $P_k = a + b\rho_{-1}X_k - bZ_k$.

For assets j, k outside the index, the result is almost immediate, since the synthetic and actual assets coincide, and so

$$\tilde{P}_j - \tilde{P}_k = b\rho_{-1}\left(\tilde{X}_j - \tilde{X}_k\right) - b\left(\tilde{Z}_j - \tilde{Z}_k\right), \tag{A-41}$$

and so

$$\operatorname{var}(\tilde{X}_j - \tilde{X}_k | \tilde{P}_j - \tilde{P}_k)^{-1} = \frac{1}{2}\tau_X + \frac{1}{2}\rho_{-1}^2\tau_Z,$$

establishing the result.

For assets j, k covered by the index, synthetic and actual assets differ, and so an additional step is required. The key step is to establish that the relative payoff of actual assets 1 and 2 (and hence, by symmetry, any pair of assets covered by the index) equals the linear combination payoffs of a set of synthetic assets that *does not* include the index asset, specifically,

$$\sum_{i=1}^{\frac{l}{2}} X_{2i} = \frac{\sqrt{l}}{2}\left(\tilde{X}_1 - \tilde{X}_2\right). \tag{A-42}$$

We establish (A-42) below. But taking this equality as given, along with the directly analogous equalities $\sum_{i=1}^{\frac{l}{2}} Z_{2i} = \frac{\sqrt{l}}{2}\left(\tilde{Z}_1 - \tilde{Z}_2\right)$ and $\sum_{i=1}^{\frac{l}{2}} P_{2i} = \frac{\sqrt{l}}{2}\left(\tilde{P}_1 - \tilde{P}_2\right)$, we know

$$\frac{\sqrt{l}}{2}\left(\tilde{P}_1 - \tilde{P}_2\right) = \sum_{i=1}^{\frac{l}{2}} P_{2i} = b\rho_{-1}\sum_{i=1}^{\frac{l}{2}} X_{2i} - b\sum_{i=1}^{\frac{l}{2}} Z_{2i} = b\rho_{-1}\frac{\sqrt{l}}{2}\left(\tilde{X}_1 - \tilde{X}_2\right) - b\frac{\sqrt{l}}{2}\left(\tilde{Z}_1 - \tilde{Z}_2\right),$$

which coincides with (A-41), and hence establishes the result for assets 1 and 2, and hence (by symmetry) for any pair of assets j, k that are covered by the index.

It remains to establish (A-42). To do so, we show that the row vectors A_i of the matrix A satisfy

$$\sum_{i=1}^{\frac{l}{2}} A_{2i} = \frac{\sqrt{l}}{2}\left(1, -1, \mathbf{0}_{m-2}\right), \tag{A-43}$$

where $\mathbf{0}_{m-2}$ denotes a row vector of length $m-2$ in which all entries equal 0. The proof of (A-43) can be rolled into the existing inductive proof of Lemma 1, where recall that we focus on the case $m = l$, since the extension to $m > l$ is trivial. The base case is $l = 2$, and holds since $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. For the inductive step, suppose that (A-43) holds for some l; the matrix for the case $2l$ is

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} A & A \\ A & -A \end{pmatrix}.$$

Hence

$$\sum_{i=1}^{\frac{2l}{2}} B_{2i} = 2\frac{1}{\sqrt{2}}\frac{\sqrt{l}}{2}\left(1,-1,\mathbf{0}_{l-2},\mathbf{0}_l\right) = \frac{\sqrt{2l}}{2}\left(1,-1,\mathbf{0}_{2l-2}\right),$$

thereby completing the proof.

Proof of Lemma 7: By symmetry of assets covered by the index, it suffices to establish the result for the underlying asset $j=1$. From Lemma 1, the underlying asset 1 is related to synthetic assets $1,\ldots,l$ by $\tilde{X}_1 = \frac{1}{\sqrt{l}}\sum_{i=1}^{l} X_i$, with analogous equalities for $\tilde{P}_1$ and $\tilde{Z}_1$. In equilibrium, there are constants a_1, b_1, b_{-1} such that synthetic asset prices are given by $P_1 = a_1 + b_1\rho_1 X_1 - b_1 Z_1$, and for any $j = 2,\ldots,l$, by $P_j = b_{-1}\rho_{-1}X_j - b_{-1}Z_j$. Moreover, $\rho_1 \leq \rho_{-1}$, where the inequality is strict in any indexing equilibrium. Combining these observations,

$$\begin{aligned}
\tilde{P}_1 &= \frac{1}{\sqrt{l}}\sum_{i=1}^{l} P_i \\
&= \frac{1}{\sqrt{l}}\left(a_1 + b_1\rho_1 X_1 + b_{-1}\rho_{-1}\sum_{i=2}^{l} X_i - b_1 Z_1 - b_{-1}\sum_{i=2}^{l} Z_i\right) \\
&= \frac{1}{\sqrt{l}}\left(\frac{b_1\rho_1 + (l-1)\,b_{-1}\rho_{-1}}{l}\sum_{i=1}^{l} X_i + \frac{l-1}{l}\left(b_1\rho_1 - b_{-1}\rho_{-1}\right)X_1 - \frac{b_1\rho_1 - b_{-1}\rho_{-1}}{l}\sum_{i=2}^{l} X_i\right) \\
&+ \frac{1}{\sqrt{l}}\left(a_1 - \frac{b_1 + (l-1)\,b_{-1}}{l}\sum_{i=1}^{l} Z_i - \frac{l-1}{l}\left(b_1 - b_{-1}\right)Z_1 + \frac{b_1 - b_{-1}}{l}\sum_{i=2}^{l} Z_i\right) \\
&= \frac{1}{\sqrt{l}}\left(\frac{b_1\rho_1 + (l-1)\,b_{-1}\rho_{-1}}{l}\sum_{i=1}^{l} X_i + \frac{b_1\rho_1 - b_{-1}\rho_{-1}}{l}\left((l-1)\,X_1 - \sum_{i=2}^{l} X_i\right)\right) \\
&+ \frac{1}{\sqrt{l}}\left(a_1 - \frac{b_1 + (l-1)\,b_{-1}}{l}\sum_{i=1}^{l} Z_i - \frac{b_1 - b_{-1}}{l}\left((l-1)\,Z_1 - \sum_{i=2}^{l} Z_i\right)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\operatorname{cov}\left(\sum_{i=1}^{l} X_i, (l-1)\,X_1 - \sum_{i=2}^{l} X_i\right) &= 0 \\
\operatorname{cov}\left(\sum_{i=1}^{l} Z_i, (l-1)\,Z_1 - \sum_{i=2}^{l} Z_i\right) &= 0.
\end{aligned}$$

Hence

$$\begin{aligned}\tilde{P}_1 &= \frac{b_1\rho_1 + (l-1)\,b_{-1}\rho_{-1}}{l}\tilde{X}_1 + \frac{b_1\rho_1 - b_{-1}\rho_{-1}}{l}\frac{1}{\sqrt{l}}\left((l-1)\,X_1 - \sum_{i=2}^{l} X_i\right) \\ &+ \frac{a_1}{\sqrt{l}} - \frac{b_1 + (l-1)\,b_{-1}}{l}\tilde{Z}_1 - \frac{b_1 - b_{-1}}{l}\frac{1}{\sqrt{l}}\left((l-1)\,Z_1 - \sum_{i=2}^{l} Z_i\right), \qquad \text{(A-44)}\end{aligned}$$

where $\tilde{X}_1,\ \tilde{Z}_1,\ \left((l-1)\,X_1 - \sum_{i=2}^{l} X_i\right)$ and $\left((l-1)\,Z_1 - \sum_{i=2}^{l} Z_i\right)$ are mutually independent. Consequently, since $\rho_1 < \rho_{-1}$,

$$\mathrm{var}\left(\tilde{X}_1|\tilde{P}_1\right)^{-1} < \tau_X + \left(\frac{b_1\rho_1 + (l-1)\,b_{-1}\rho_{-1}}{b_1 + (l-1)\,b_{-1}}\right)^2 \tau_Z < \tau_X + \rho_{-1}^2\tau_Z = \mathrm{var}\left(\tilde{X}_k|\tilde{P}_k\right)^{-1},$$

completing the proof.

Proof of Lemma 9: From (34), Lemma 4, and (A-30),

$$\mathbb{E}\left[X_1 - P_1|P_1\right] = \frac{S_1 + \mathbb{E}\left[Z_1|P_1\right]}{\frac{1}{\gamma}\frac{1}{n_1}\int_0^{n_1}\frac{1}{\mathrm{var}(X_1|Y_{i1},e_{i1},P_1)}di} = -\frac{S_1 + \mathbb{E}\left[Z_1|P_1\right]}{\frac{1}{n_1}\int_0^{n_1}\frac{\partial\theta_{i1}}{\partial P_1}di} = -\frac{S_1 + \mathbb{E}\left[Z_1|P_1\right]}{1 + \frac{\rho_1}{\gamma}\tau_Z}\frac{\partial P_1}{\partial Z_1}.$$

Note that

$$\frac{\partial}{\partial P_1}\mathbb{E}\left[Z_1|P_1\right] = \frac{\mathrm{cov}\left(Z_1, P_1\right)}{\mathrm{var}\left(P_1\right)} = \frac{\mathrm{cov}\left(Z_1, P_1\right)}{\mathrm{var}\left(Z_1\right)}\frac{\mathrm{var}\left(Z_1\right)}{\mathrm{var}\left(P_1\right)} = \frac{\partial P_1}{\partial Z_1}\frac{\mathrm{var}\left(Z_1\right)}{\left(\frac{\partial P_1}{\partial X_1}\right)^2\mathrm{var}\left(X_1\right) + \left(\frac{\partial P_1}{\partial Z_1}\right)^2\mathrm{var}\left(Z_1\right)}.$$

Hence (and using the fact that $\frac{\partial P_1}{\partial Z_1}$ is independent of Z_1)

$$\frac{\partial}{\partial P_1}\mathbb{E}\left[X_1 - P_1|P_1\right] = -\frac{1}{\left(1 + \frac{\rho_1}{\gamma}\tau_Z\right)\left(\rho_1^2\frac{\mathrm{var}(X_1)}{\mathrm{var}(Z_1)} + 1\right)},$$

completing the proof.

Appendix B

For notational transparency, we consider a single-asset version of our economy, and omit all asset subscripts.

Proposition B-1 *Consider the benchmark economy described in subsection 3.1, in which agents do not possess any private information about the asset's cash flow X, but instead all observe a public signal of the form $Y = X + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \tau_\epsilon^{-1})$. In such a setting, each agent's expected utility is decreasing in the precision of the public signal, τ_ϵ.*

Proof: Agent i's terminal wealth is

$$W_i = e_i P + (\theta_i + e_i)(X - P),$$

and he optimally chooses the portfolio

$$\theta_i + e_i = \frac{1}{\gamma}\frac{\mathbb{E}[X|Y] - P}{\mathrm{var}(X|Y)}.$$

So agent i's expected utility at the trading stage is

$$\begin{aligned}
\mathbb{E}[-\exp(-\gamma W_i)|Y,P] &= \mathbb{E}\left[-\exp\left(-\gamma e_i P - \frac{\mathbb{E}[X|Y] - P}{\mathrm{var}(X|Y)}(X - P)\right)|Y,P\right] \\
&= -\exp\left(-\gamma e_i P - \frac{(\mathbb{E}[X|Y] - P)^2}{\mathrm{var}(X|Y)} + \frac{1}{2}\frac{(\mathbb{E}[X|Y] - P)^2}{\mathrm{var}(X|Y)}\right) \\
&= -\exp\left(-\gamma e_i P - \frac{1}{2}\frac{(\mathbb{E}[X|Y] - P)^2}{\mathrm{var}(X|Y)}\right).
\end{aligned}$$

We evaluate

$$\mathbb{E}\left[-\exp\left(-\gamma e_i P - \frac{1}{2}\frac{(\mathbb{E}[X|Y] - P)^2}{\mathrm{var}(X|Y)}\right)|e_i\right]. \tag{B-1}$$

Expanding, this expression equals

$$\mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}[X|Y] + \gamma e_i(\mathbb{E}[X|Y] - P) - \frac{1}{2}\frac{(\mathbb{E}[X|Y] - P)^2}{\mathrm{var}(X|Y)}\right)|e_i\right].$$

By market clearing,

$$\frac{1}{\gamma}\frac{\mathbb{E}[X|Y] - P}{\mathrm{var}(X|Y)} = S + Z,$$

i.e.,

$$\mathbb{E}\left[X|Y\right] - P = \gamma \mathrm{var}\left(X|Y\right)\left(S+Z\right),$$

and so (B-1) equals

$$\mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}\left[X|Y\right] + \frac{\gamma^2 \mathrm{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right) - \left(S+Z\right)^2\right)\right)|e_i\right].$$

Moreover,

$$\mathbb{E}\left[X|Y\right] = \frac{\tau_X \mathbb{E}\left[X\right] + \tau_\epsilon Y}{\tau_X + \tau_\epsilon} = \frac{\tau_\epsilon^{-1}\mathbb{E}\left[X\right] + \tau_X^{-1} Y}{\tau_X^{-1} + \tau_\epsilon^{-1}} = \frac{\left(\mathrm{var}\left(Y\right) - \mathrm{var}\left(X\right)\right)\mathbb{E}\left[X\right] + \mathrm{var}\left(X\right) Y}{\mathrm{var}\left(Y\right)}.$$

Hence (B-1) equals

$$\mathbb{E}\left[\exp\left(-\gamma e_i \mathbb{E}\left[X\right] + \frac{\gamma^2 e_i^2}{2}\frac{\mathrm{var}\left(X\right)^2}{\mathrm{var}\left(Y\right)} + \frac{\gamma^2 \mathrm{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right) - \left(S+Z\right)^2\right)\right)|e_i\right].$$

By the law of total variance,

$$\mathrm{var}\left(X\right) = \mathrm{var}\left(X|Y\right) + \mathrm{var}\left(E\left[X|Y\right]\right) = \mathrm{var}\left(X|Y\right) + \frac{\mathrm{var}\left(X\right)^2}{\mathrm{var}\left(Y\right)}.$$

So (B-1) equals

$$\begin{aligned} & \mathbb{E}\left[-\exp\left(-\gamma e_i \mathbb{E}\left[X\right] + \frac{\gamma^2 e_i^2}{2}\mathrm{var}\left(X\right) + \frac{\gamma^2 \mathrm{var}\left(X|Y\right)}{2}\left(2e_i\left(S+Z\right) - \left(S+Z\right)^2 - e_i^2\right)\right)|e_i\right] \\ = \; & \mathbb{E}\left[-\exp\left(-\gamma e_i X - \frac{\gamma^2 \mathrm{var}\left(X|Y\right)}{2}\left(e_i - \left(S+Z\right)\right)^2\right)|e_i\right]. \end{aligned}$$

This expression is increasing in $\mathrm{var}\left(X|Y\right)$, completing the proof.