Technical appendix to "Regulating exclusion from financial markets"
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This document contains several proofs that are referred to, but omitted from, the published version of our paper "Regulating exclusion from financial markets."

The existence of a solution to the basic problem of Section 2

In Section 2 we claim that the constraint set associated with constraints (9) - (11) is closed and bounded. To see this, note that constraint (11) can be rewritten

$$P_{t'} - L_{t'} \leq \prod_{\tilde{s}=t}^{t'-1} R_{\tilde{s}} W + \frac{1}{\prod_{\tilde{s}=t'}^{T-1} R_{\tilde{s}}} \left(\sum_{s=t}^{T-1} \left(\prod_{\tilde{s}=s}^{T-1} R_{\tilde{s}} \right) (L_s - P_s) - \sum_{s=t'}^{T} \left(\prod_{\tilde{s}=s}^{T-1} R_{\tilde{s}} \right) (L_s - P_s) \right)$$

which from (9) and (10) implies

$$P_{t'} - L_{t'} \leq \prod_{\tilde{s}=t}^{t'-1} R_{\tilde{s}} W + \frac{1}{\prod_{\tilde{s}=t'}^{T-1} R_{\tilde{s}}} \left(\prod_{\tilde{s}=t}^{T-1} R_{\tilde{s}} v - \left(\prod_{\tilde{s}=t'}^{T-1} R_{\tilde{s}} \right) L_{t'} \right)$$

That is,

$$P_{t'} \leq \prod_{\tilde{s}=t}^{t'-1} R_{\tilde{s}} (W + v)$$

So given any values of W, v, the choice of $\{P_{t'} : t' \geq t\}$ is certainly bounded. Conditions (9) and (10) by themselves imply

$$\left(\prod_{\tilde{s}=t'}^{T-1} R_{\tilde{s}} \right) L_{t'} \leq \left(\prod_{\tilde{s}=t}^{T-1} R_{\tilde{s}} \right) v - \sum_{s=t}^{t'-1} \left(\prod_{\tilde{s}=t}^{t-1} R_{\tilde{s}} \right) (L_s - P_s)$$

It is then straightforward to iteratively establish that the choices $\{L_{t'} : t' \geq t\}$ are also bounded.

Proof of Proposition 1

To establish the result, we first compute the solution to the planning problem of maximizing the objective (8) subject to the constraints (9), (10) and (11). In the main text we showed this problem can be written recursively in equations (3), (4), (5), (6), and (7). We can further simplify the problem as,

$$V_t^M (W, v) = \max_{P \in [0, W+L], L \in [0, v]} -(L - P) + \frac{1}{r} V_{t+1}^M (R_t (W + L - P), R_t (v - (L - P))) \tag{21}$$

Recall that $V_t^M(W,v)$ is the maximal present value of profits from date t onwards attainable by the coalition of all M banks.

Trivially in the final period T the bank's value function is just $V_T^M(W,v) = -v$. From the linearity of the problem and the form taken by $V_T^M(W,v)$, we guess (and verify below) that for all $t < T$ the value function $V_t^M(W,v)$ is linear in W and v with coefficients that sum to 1,

$$V_t^M(W,v) = \alpha_t W - (1-\alpha_t)v \tag{22}$$

Thus $\alpha_T = 0$, and

$$V_t^M(W,v) = \max_{P\in[0,W+L],L\in[0,v]} (L-P)(\rho_t - 1) + (W\alpha_{t+1} - v(1-\alpha_{t+1}))\rho_t$$

In any payment period t, the bank's rate of return is higher than the borrower's. In these periods the borrower is best off transferring all his resources to the bank, so that the bank can invest them at the higher rate $r > R_t$. Formally, since $\rho_t < 1$ we must have $P - L = W$. Without loss, we can set $L = 0$ and $P = W$, i.e. no new loan, and borrower transfers all his wealth to the bank. In this case,

$$V_t^M(W,v) = W(1-(1-\alpha_{t+1})\rho_t) - v(1-\alpha_{t+1})\rho_t$$

so that

$$\alpha_t = 1 - (1-\alpha_{t+1})\rho_t \tag{23}$$

Note also that $W' = 0$ and $v' = R_t(v+W)$ — the borrower now has no wealth, but the amount that the bank must transfer to the borrower has increased from v to $R_t(v+W)$.

In any investment period t, the bank's rate of return is lower than the borrower's, so there is potentially scope for lending. Since $R_t \geq r$, then $\rho_t \geq 1$ and so $P = 0$ and $L = v$ is optimal. In this case,

$$V_t^M(W,v) = W\alpha_{t+1}\rho_t - v(1-\alpha_{t+1}\rho_t)$$

so that

$$\alpha_t = \alpha_{t+1}\rho_t \tag{24}$$

Note also that $W' = R_t(v+W)$ and $v' = 0$ — the bank no longer "owes" the borrower anything, and makes no further transfers until after the borrower has made some payments to the bank

We now turn to a description of the actual payments. First note that if t is an investment period that is followed by another investment period, since $v_{t+1} = 0$ we know that no payments are made in period $t+1$. Likewise, if t is a payment period that is followed by a payment period, since $W_{t+1} = 0$ we again know that no payments are made in period

$t+1$. Thus the funds are transferred between the bank and borrower only in some subset of periods $t_0 = 0, t_1, t_2, \ldots, t_\tau$ where t_{i+1} is an investment (respectively, payment) period if and only if t_i and $t_{i+1}-1$ are payment (respectively, investment) periods. Note that since T is an investment period it follows that t_τ must be an investment period.

The resulting payments are as follows. At $t_0 = 0$ the bank makes an initial loan of $L_0 = v_0$. In the first payment period t_1, the borrower repays the bank all his wealth, $P_{t_1} = W_{t_1} = (W_0 + L_0) \prod_{s=t_0}^{t_1-1} R_s$. In the next investment period, t_2, the bank makes a new loan of $L_{t_2} = v_{t_2} = P_{t_1} \prod_{s=t_1}^{t_2-1} R_s$. The cycle then continues until at t_τ the bank makes the last "loan", $L_{t_\tau} = P_{t_{\tau-1}} \prod_{s=t_{\tau-1}}^{t_\tau-1} R_s$.

Finally, the maximal loan size and final borrower consumption can easily be determined as follows. For the banks to be collectively break-even we must certainly have

$$V_0 (W_0, v_0) = \alpha_0 W_0 - (1-\alpha_0) v_0 \geq 0$$

and so

$$v_0 = L_0 \leq \frac{W_0 \alpha_0}{1-\alpha_0}$$

The borrower's maximal final consumption is then

$$\left(W_0 + \frac{W_0 \alpha_0}{1-\alpha_0} \right) \prod_{t=0}^{T-1} R_t = \frac{W_0}{1-\alpha_0} \prod_{t=0}^{T-1} R_t$$

Proof of Proposition 3

Proposition 3 states that the enforcement rules $\mathcal{B}_{KLK}$, $\mathcal{B}_{excl}$ and $\mathcal{B}_{DD}$ all support the constrained optimum as an equilibrium. In the text we established the result for the last two of the three cases. Here we give a formal proof for the case in which the $\mathcal{B}_{KLK}$ is in effect.

We establish that the following lending policies and borrower payments constitute an equilibrium, given the enforcement rule $\mathcal{B}_{KLK}$: (a) Bank 1's lending policy $\mathcal{L}^1$ is $l_t^1 \equiv L_t^*$, (b) Every other bank $m \neq 1$ offers the lending policy $\mathcal{L}^m$, $l_t^m (\mathbf{P}_0, \ldots, \mathbf{P}_{t-1}) \equiv r P_{t-1}^m$ (i.e. take deposits at rate r), (c) The borrower repays $P_t^1 = P_t^*$ and $P_t^m = 0$. By construction, in this equilibrium all banks make zero profits and the borrower's final consumption is $W_0 \prod_{t=0}^{T-1} \max \{r, R_t\}$.

To show that we have actually described an equilibrium, start by noting that since the payments P_t^* were defined to be feasible and incentive compatible given loans L_t^* and the threat of full exclusion, the borrower's payment strategy is certainly a best response given lending policies $\mathcal{L}^1$ and $\mathcal{L}^m$ and the enforcement rule $\mathcal{B}_{KLK}$.

Next, we claim that $\mathcal{L}^1$ is a best response to $\mathcal{L}^m$. For suppose to the contrary that there exists $\hat{\mathcal{L}}^1$ delivering strictly positive profits to bank 1. Let $\hat{L}_t^1$ and $\hat{P}_t^1$ be the equilibrium loan payments under this deviation. The loan payments $\hat{L}_t^1$ must differ from the original loan payments L_t^* at at least one date. Let τ be the first such date. If $\hat{L}_\tau^1 < L_\tau^*$ then at the first payment period to follow τ, the borrower has insufficient resources to repay the bank P_t^*. But then date τ is the last period in which any transfer occurs between the borrower and the bank. However, this then implies that the borrower would not have made the payment $P_{\tau'+1}^*$ where τ' is the last investment period prior to τ, since be definition $L_t^* = \prod_{s=\tau'+1}^{\tau-1} R_s P_{\tau'+1}^*$. Since any payment less than $P_{\tau'+1}^*$ leads to full exclusion, the borrower simply pays 0 at this date. On the other hand, if $\hat{L}_\tau^1 > L_\tau^*$ then the borrower's incentive constraint is now violated at the first payment period to follow τ. In either case, bank 1 is left with negative profits.

Finally, we claim that $\mathcal{L}^m$ is a best response to $\mathcal{L}^1$ for every $m \neq 1$. For under any strictly profitable deviation, bank m's deviation must involve him making a positive payment to the borrower before receiving any payments from the borrower, since the borrower must pay all his wealth to bank 1 in each payment period.[25] By the same argument as above, the borrower now prefers to default on everyone over repaying both banks 1 and m. Moreover, if he just defaults on bank 1, his payment to bank m is seized. So regardless of whether he defaults on both banks, just on bank m, or just on bank 1, bank m will receive no payment after the initial loan in period τ. Thus his deviation cannot have been profitable.

Proof of Proposition 5

Proof outline

We proceed as follows:

1. We characterize the payoffs of an equilibrium of the subgame starting at date $T-1$ in which the borrower is indebted to only one of the banks (without loss, bank 1).
2. Proceeding inductively, we then characterize the payoffs and payments of an equilibrium of the subgame starting at any prior date $t < T-1$, under the assumption that *only bank* 1 *is present*.
3. Finally, we show that the equilibrium constructed is still an equilibrium when remaining $M-1$ banks are present at dates $t < T-1$.

[25] Technically there is also the possibility that the borrower could make a payment to the deviating bank m in an investment period. But then bank m would have to pay an interest rate of $R_t > r$ on this deposit to avoid the borrower defaulting at the next payment period.

Step 1: The subgame at date $t = T - 1$

Lemma 4 *Suppose that at date $T-1$ the borrower has wealth W, a debt level with bank 1 of D, while $D^m_{T-1} = 0$ for all other banks $m \neq 1$. Let $\gamma_{T-1} = 1 - \rho_{T-1}$, as defined in the statement of Proposition 4. Then there is a subgame perfect equilibrium in which the final consumption of the borrower is given by*

$$U^B_{T-1}(W, D) = \begin{cases} Wr & \text{if } D < 0 \\ (W - D)\, r & \text{if } D \in [0, W\gamma_{T-1}] \\ WR_{T-1} & \text{if } D > W\gamma_{T-1} \end{cases}$$

while the final profits of bank 1 (in date T terms) are given by

$$U^1_{T-1}(W, D) = \begin{cases} -Dr & \text{if } D < 0 \\ 0 & \text{if } D \in [0, W\gamma_{T-1}] \\ (-D + W\gamma_{T-1})\, r & \text{if } D > W\gamma_{T-1} \end{cases}$$

All of the remaining banks $m \neq 1$ have a final profit of zero.

Proof: If $D < 0$ then it is straightforward to show that it is an equilibrium for all banks $m \in M$ to set $L^m_{T-1} = 0$ and $l^m_T(\mathbf{P}_{T-1}) = rP^m_{T-1}$ (i.e. offer to accept savings at a rate r). The payoffs are then immediate.

Suppose on the other hand that $D \geq 0$. Then it is an equilibrium for banks $m \neq 1$ to set $L^m_{T-1} = 0$ and $l^m_T(\mathbf{P}_{T-1}) = rP^m_{T-1}$, while bank 1 sets $L^1_{T-1} = 0$ and

$$l^1_T(\mathbf{P}_{T-1}) = \begin{cases} (W - D)\, r & \text{if } P^1_{T-1} = W \text{ and } D \in [0, W\gamma_{T-1}] \\ WR_{T-1} & \text{if } P^1_{T-1} = W \text{ and } D > W\gamma_{T-1} \\ 0 & \text{otherwise} \end{cases}$$

and for the borrower to respond by paying all his wealth to bank 1, i.e. $\mathbf{P}_{T-1} = (W, 0, \ldots, 0)$.

There are two cases to deal with in verifying that $\mathbf{P}_{T-1} = (W, 0, \ldots, 0)$ is indeed a best response for the borrower:

Case (A): $D \in [0, W\gamma_{T-1}]$. If he sets $P^1_{T-1} = W$ he gets $(W - D)\, r$. If he sets $P^1_{T-1} \in [D, W)$ he can get at most $\left(W - P^1_{T-1}\right) r$, which is at least weakly worse. If he sets $P^1_{T-1} \in [0, D)$ the debt-default rule $\mathcal{B}_{DD}$ prevents him from transferring any funds to any bank $m \neq 1$ and so he gets $\left(W - P^1_{T-1}\right) R_{T-1} \leq WR_{T-1} \leq (W - D)\, r$, where the second inequality follows from the fact that $D \leq W\gamma_{T-1}$.

Case (B): $D > W\gamma_{T-1}$. If he sets $P^1_{T-1} = W$ he gets WR_{T-1}. If he sets $P^1_{T-1} \in [D, W)$ he can get at most $\left(W - P^1_{T-1}\right) r \leq (W - D)\, r \leq WR_{T-1}$ where the second inequality follows from the fact that $D \leq W\gamma_{T-1}$. If he sets $P^1_{T-1} \in [0, D)$ the debt-default rule $\mathcal{B}_{DD}$ prevents him from transferring any funds to any bank $m \neq 1$ and so he gets $\left(W - P^1_{T-1}\right) R_{T-1} \leq W R_{T-1}$.

Given the lending policies of the banks $m \neq 1$, bank 1's lending policy is clearly a best response. Likewise, the lending policies of banks $m \neq 1$ are a best response to bank 1's lending policy. The payoffs stated in the Lemma are then immediate. **QED**

Step 2: Periods $t < T - 1$

We now proceed to construct a subgame perfect equilibrium of the game in which prior to period $T-1$ only bank 1 is present, and from period $T-1$ onwards the borrower and the banks play the equilibrium described in Lemma 4.

Let the equilibrium payoffs at date t be denoted by $U^B_t(W, D)$ and $U^1_t(W, D)$ for the borrower and bank 1 respectively, where D is the borrower's level of indebtedness at that date to bank 1 and W is his wealth level. From Lemma 4 we guess (and will verify) that $U^B_t(W, D)$ and $U^1_t(W, D)$ take the forms

$$\begin{aligned} U^B_t(W, D) &= \begin{cases} W a_{1,t} & \text{if } D < 0 \\ (W - D)\, a_{2,t} & \text{if } D \in [0, W\delta_t] \\ W a_{3,t} & \text{if } D > W\delta_t \end{cases} \\ U^1_t(W, D) &= \begin{cases} -D a_{4,t} & \text{if } D < 0 \\ 0 & \text{if } D \in [0, W\delta_t] \\ (-D + W a_{5,t})\, r^{T-t} & \text{if } D > W\delta_t \end{cases} \end{aligned} \tag{25}$$

Investment periods

Lemma 5 *Suppose we are in an investment period $t < T-1$ (i.e. $R_t > r$) and that the payoff functions U^B_{t+1} and U^1_{t+1} are of the form given in (25) with coefficients satisfying*

$$a_{1,t+1} \leq a_{2,t+1} \tag{26}$$
$$(1 - \delta_{t+1})\, a_{2,t+1} \leq a_{3,t+1} \tag{27}$$
$$a_{5,t+1} \leq \delta_{t+1} \tag{28}$$
$$\delta_{t+1}\rho_t < 1 \tag{29}$$

Then U^B_t and U^1_{t+1} are also of the form (25) with coefficients

$$\begin{aligned} a_{1,t} &= R_t a_{1,t+1} \\ a_{2,t} &= R_t \frac{1 - \delta_{t+1}}{1 - \rho_t \delta_{t+1}} a_{2,t+1} \end{aligned}$$

$$\begin{aligned}
a_{3,t} &= R_t a_{3,t+1} \\
a_{4,t} &= r a_{4,t+1} \\
a_{5,t} &= \rho_t a_{5,t} \\
\delta_t &= \rho_t \delta_{t+1}
\end{aligned}$$

Bank 1 makes an loan of $\frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}}$ provided that the debt level D falls between 0 and $W\delta_t$, and otherwise makes no loan. The borrower does not repay anything in this period.

Proof: Given W, D, L, P,

$$\begin{aligned}
W_{t+1} &= (W + L - P) R_t \\
D_{t+1} &= (D + L - P) r
\end{aligned}$$

and so

$$\begin{aligned}
W_{t+1} - D_{t+1} = W R_t - D r + (L - P)(R_t - r) \\
D_{t+1} \leq W_{t+1}\delta_{t+1} \Longleftrightarrow L - P \leq \frac{W\rho_t\delta_{t+1} - D}{1 - \rho_t\delta_{t+1}} \\
D_{t+1} \geq 0 \Longleftrightarrow L - P \geq -D
\end{aligned}$$

where the second equivalence follows from condition (29). Substituting into the expressions for (25)

$$\begin{aligned}
U^B_{t+1}(W, D) &= \begin{cases} (W + L - P) R_t a_{1,t+1} & \text{if } L - P < -D \\ (W R_t - D r + (L - P)(R_t - r)) a_{2,t+1} & \text{if } -D \leq L - P \leq \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}} \\ (W + L - P) R_t a_{3,t+1} & \text{if } L - P > \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}} \end{cases} \\
U^1_{t+1}(W, D) &= \begin{cases} -(D + L - P) r a_{4,t+1} & \text{if } L - P \leq -D \\ 0 & \text{if } -D < L - P \leq \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}} \\ \left(-(D + L - P) r \right. \\ \left. \quad + (W + L - P) R_t a_{5,t+1}\right) r^{T-(t+1)} & \text{if } L - P > \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}} \end{cases}
\end{aligned} \tag{30}$$

Note that when $L - P = -D$,

$$W R_t - D r + (L - P)(R_t - r) = (W - D) R_t = (W + L - P) R_t$$

while when $L - P = \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}}$

$$\begin{aligned}
W R_t - D r + (L - P)(R_t - r) &= (W - D)\frac{R_t(1 - \delta_{t+1})}{1 - \rho_t\delta_{t+1}} \\
&= (1 - \delta_{t+1})(W + L - P) R_t
\end{aligned}$$

It follows from $R_t > r$ and conditions (26) and (27) that the borrower's utility is a strictly decreasing function of the payment P. Thus $P = 0$.

Next, we turn to the lender's payment L. If $D < 0$, then the lender cannot do better than set $L = 0$. On the other hand, if $D \geq 0$ then the only way for the lender to achieve non-negative utility is to choose L such that $0 \leq L \leq \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}}$. Since the borrower's utility is an increasing function of L (by the same argument that it is a decreasing function of P) we have $L = \frac{W\rho_t\delta_{t+1}-D}{1-\rho_t\delta_{t+1}}$ provided this is non-negative, i.e. provided $D \leq W\rho_t\delta_{t+1}$, and $L = 0$ otherwise. Substituting into our expressions for U_{t+1}^B and U_{t+1}^1 we obtain

$$U_t^B(W,D) = \begin{cases} WR_t a_{1,t+1} & \text{if } D < 0 \\ (W-D)\frac{R_t(1-\delta_{t+1})}{1-\rho_t\delta_{t+1}} a_{2,t+1} & \text{if } D \in [0, W\rho_t\delta_{t+1}] \\ WR_t a_{3,t+1} & \text{if } D > W\rho_t\delta_{t+1} \end{cases}$$

$$U_t^1(W,D) = \begin{cases} -Dra_{4,t+1} & \text{if } D < 0 \\ 0 & \text{if } D \in [0, W\rho_t\delta_{t+1}] \\ (-D + W\rho_t a_{5,t+1})\, r^{T-t} & \text{if } D > W\rho_t\delta_{t+1} \end{cases}$$

which completes the proof. **QED**

Payment periods

Lemma 6 *Suppose we are in a payment period $t < T$ (i.e. $R_t \leq r$) and that the payoff functions U_{t+1}^B and U_{t+1}^1 are of the form given in (25) with coefficients satisfying*

$$\frac{a_{2,t+1}}{a_{3,t+1}} \geq \frac{1}{1-\rho_t\delta_{t+1}} \tag{31}$$

$$a_{5,t+1} \leq \delta_{t+1} \tag{32}$$

$$\delta_{t+1}\rho_t < 1 \tag{33}$$

Then U_t^B and U_t^1 are also of the form (25) with coefficients

$$\begin{aligned} a_{1,t} &= R_t a_{1,t+1} \\ a_{2,t} &= R_t a_{2,t+1} \\ a_{3,t} &= R_t a_{3,t+1} \\ a_{4,t} &= r a_{4,t+1} \\ a_{5,t} &= \rho_t a_{5,t+1} \\ \delta_t &= 1 - \frac{a_{3,t+1}}{a_{2,t+1}} \end{aligned}$$

Bank 1 does not make any loan in this period. The borrower repays his debt D in full provided it between 0 and $W\delta_t$, and does not repay anything otherwise.

Proof: Given W, D, P, L we obtain exactly the same characterization of U^B_{t+1} and U^1_{t+1} as in for investment periods (see (30) in the proof of Lemma 5). We first analyze the choice of the borrower's payment ignoring the wealth constraint $P \leq W + L$, and then show it does not bind.

Clearly the borrower will never choose to make a strictly positive payment P such that $L - P < -D$ or $L - P > \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}}$, since in this range a lower payment would be strictly better. On the other hand, since $r \geq R_t$ the borrower weakly prefers $P = D + L$ to all other payments P for which $-D \leq L - P \leq \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}}$. So (ignoring the wealth constraint) the borrower's choice reduces to one between $P = 0$ and $P = D + L$. Choosing $P = L$ leads to

$$U^B_{t+1}(W, D) = \begin{cases} (W + L) R_t a_{1,t+1} & \text{if } L < -D \\ (WR_t - Dr + L(R_t - r)) a_{2,t+1} & \text{if } -D \leq L \leq \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}} \\ (W + L) R_t a_{3,t+1} & \text{if } L > \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}} \end{cases}$$

while $P = L + D$ gives

$$U^B_{t+1}(W, D) = (W - D) R_t a_{2,t+1}$$

Now, $P = L + D$ is only a feasible choice if it is non-negative, so if $L < -D$ the borrower chooses $P = 0$. For $-D < L \leq \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}}$ then $P = L + D$ is trivially the better choice. Finally, for $L > \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}}$ then $P = L + D$ is better if

$$(W - D) R_t a_{2,t+1} \geq (W + L) R_t a_{3,t+1}$$

or equivalently

$$L \leq (W - D) \frac{a_{2,t+1}}{a_{3,t+1}} - W \tag{34}$$

When $W - D \geq 0$, then given condition (31) the inequality

$$L \leq \frac{W\rho_t\delta_{t+1} - D}{1 - \rho_t\delta_{t+1}} = \frac{(W - D)}{1 - \rho_t\delta_{t+1}} - W$$

holds whenever (34) does. On the other hand, if $W - D < 0$ then there is no $L \geq 0$ satisfying $-D < L \leq \frac{W\rho_t\delta_{t+1} - D}{1-\rho_t\delta_{t+1}}$ or (34). We have now almost established that $P = L + D$ whenever $L \geq 0$ and

$$-D \leq L \leq (W - D) \frac{a_{2,t+1}}{a_{3,t+1}} - W \tag{35}$$

and $P = 0$ otherwise. It remains only to check that the wealth constraint is satisfied. If $P = 0$ there is nothing to check, while we have just argued that $P = L + D$ is only chosen if $W \geq D$, in which case the wealth constraint $P \leq W + L$ is satisfied.

We now turn to the bank's choice of L. Given L, his interim utility $u_t^1(W,D,L)$ is

$$u_t^1(W,D)=\begin{cases}-(D+L)\,ra_{4,t+1} & \text{if } L<-D\\ 0 & \text{if } -D\le L\le (W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W\\ \left(-(D+L)\,r\right. & \\ \left.+(W+L)\,R_t a_{5,t+1}\right)r^{T-(t+1)} & \text{if } L>(W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W\end{cases}$$

Now, from conditions (33) and (32) $-(D+L)\,r+(W+L)\,R_t a_{5,t+1}$ is negative if and only if

$$L>\frac{W-D}{1-\rho_t a_{5,t+1}}-W$$

which by conditions (32) and (31) holds whenever $L>(W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W$. So if $D<0$, the lender's best choice is $L=0$, while if $D\ge 0$ the lender cannot do better than set $L=0$ if $(W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W\ge 0$ (note that the borrower does not care about the choice of L between 0 and $(W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W$, since he will just pay $P=D+L$). Finally, if $(W-D)\frac{a_{2,t+1}}{a_{3,t+1}}-W<0$ then $L=0$ is the lender's best choice by conditions (33) and (32) again. Substituting into our expressions for U_{t+1}^B and U_{t+1}^1 we obtain

$$\begin{aligned}U_t^B(W,D) &= \begin{cases}WR_t a_{1,t+1} & \text{if } D<0\\ (W-D)\,R_t a_{2,t+1} & \text{if } D\in\left[0,W\left(1-\frac{a_{3,t+1}}{a_{2,t+1}}\right)\right]\\ WR_t a_{3,t+1} & \text{if } D>W\left(1-\frac{a_{3,t+1}}{a_{2,t+1}}\right)\end{cases}\\ U_t^1(W,D) &= \begin{cases}-Dra_{4,t+1} & \text{if } D<0\\ 0 & \text{if } D\in\left[0,W\left(1-\frac{a_{3,t+1}}{a_{2,t+1}}\right)\right]\\ \left(-D+W\rho_t a_{5,t+1}\right)r^{T-t} & \text{if } D>W\left(1-\frac{a_{3,t+1}}{a_{2,t+1}}\right)\end{cases}\end{aligned}$$

which completes the proof. **QED**

Verifying the form of the payoff functions $U_t^B(W,D)$ and $U_t^1(W,D)$ Next, we confirm that the conditions needed to apply Lemmas 5 and 6 are in fact satisfied:

Lemma 7 *Suppose that for $t=0,1,\ldots,T-1$ the coefficients $a_{1,t},a_{2,t},a_{3,t},a_{4,t},a_{5,t},\delta_t$ are defined iteratively by*

$$\left(a_{1,T-1},a_{2,T-1},a_{3,T-1},a_{4,T-1},a_{5,T-1},\delta_{T-1}\right)=\left(r,r,R_{T-1},r,1-\rho_{T-1},1-\rho_{T-1}\right)\quad(36)$$

and

$$a_{1,t} = R_t a_{1,t+1}\quad(37)$$

$$a_{2,t} = \begin{cases}R_t\frac{1-\delta_{t+1}}{1-\rho_t\delta_{t+1}}a_{2,t+1} & \text{if } R_t>r\\ R_t a_{2,t+1} & \text{if } R_t<r\end{cases}\quad(38)$$

$$a_{3,t} = R_t a_{3,t+1} \tag{39}$$
$$a_{4,t} = r a_{4,t+1} \tag{40}$$
$$a_{5,t} = \rho_t a_{5,t+1} \tag{41}$$
$$\delta_t = \begin{cases} \rho_t \delta_{t+1} & \text{if } R_t > r \\ 1 - \frac{a_{3,t+1}}{a_{2,t+1}} & \text{if } R_t < r \end{cases} \tag{42}$$

Then for any $t = 1, \ldots, T-1$

$$\delta_t = \gamma_t = (1 - \rho_{T-1}) \prod_{s=t}^{T-2} \max\{1, \rho_s\} \tag{43}$$
$$\delta_t \rho_{t-1} < 1 \tag{44}$$
$$a_{1,t} \leq a_{2,t} \tag{45}$$
$$a_{5,t} \leq \delta_t \tag{46}$$
$$\frac{a_{2,t}}{a_{3,t}} = \frac{1}{1 - \delta_t} \tag{47}$$

Moreover, in any payment period $t < T - 1$ *(i.e.* $R_t < r$*)*

$$\frac{a_{2,t+1}}{a_{3,t+1}} \geq \frac{1}{1 - \rho_t \delta_{t+1}} \tag{48}$$

Proof. We proceed by induction. Fix t, and suppose that the result holds for all $s > t$.

First, consider the case where t is a payment period ($R_t < r$). If $R_{t+1} > r$ then

$$\delta_t = 1 - \frac{a_{3,t+1}}{a_{2,t+1}} = 1 - \frac{a_{3,t+2}}{a_{2,t+2}} \frac{1 - \rho_{t+1}\delta_{t+2}}{1 - \delta_{t+2}} = 1 - (1 - \rho_{t+1}\delta_{t+2}) = \rho_{t+1}\delta_{t+2} = \delta_{t+1}$$

while if $R_{t+1} < r$ then

$$\delta_t = 1 - \frac{a_{3,t+1}}{a_{2,t+1}} = 1 - \frac{a_{3,t+2}}{a_{2,t+2}} = \delta_{t+1}$$

Condition (43) follows since $\rho_t < 1$ and so $\delta_t = \max\{1, \rho_t\} \delta_{t+1}$, and condition (44) is then immediate from the assumption that $\alpha_0 < 1$. Condition (45) follows trivially by induction since when $R_t < r$ we have $a_{1,t} = R_t a_{1,t+1}$ and $a_{2,t} = R_t a_{2,t+1}$. If $R_{t+1} > r$ then condition (46) follows immediately by induction. On the other hand, if $R_{t+1} < r$ then substituting in for $a_{5,t}$ and δ_t, condition (46) is equivalent to

$$\rho_t a_{5,t+1} \leq 1 - \frac{a_{3,t+1}}{a_{2,t+1}}$$

which holds since by induction $\rho_t a_{5,t+1} \leq \rho_t \delta_{t+1}$ and $1 - \frac{a_{3,t+1}}{a_{2,t+1}} \geq \rho_t \delta_{t+1}$. Finally, condition (47) holds since

$$\frac{a_{2,t}}{a_{3,t}} = \frac{a_{2,t+1}}{a_{3,t+1}} = \frac{1}{1 - \delta_t}$$

Next, turn to the case in which t is an investment period ($R_t > r$). The characterization (43) of δ_t is immediate and condition (44) again follows from the assumption that $\alpha_0 < 1$. Condition (45) holds since $\rho_t > 1$ and thus $\frac{1-\delta_{t+1}}{1-\rho_t\delta_{t+1}} > 1$, and so

$$a_{1,t} = R_t a_{1,t+1} \leq R_t \frac{1-\delta_{t+1}}{1-\rho_t\delta_{t+1}} a_{2,t+1} = a_{2,t}$$

given that $a_{1,t+1} \leq a_{2,t+1}$. Condition (46) follows trivially by induction. Finally, condition (47) holds since

$$\frac{a_{2,t}}{a_{3,t}} = \frac{1-\delta_{t+1}}{1-\rho_t\delta_{t+1}} \frac{a_{2,t+1}}{a_{3,t+1}} = \frac{1-\delta_{t+1}}{1-\rho_t\delta_{t+1}} \frac{1}{1-\delta_{t+1}} = \frac{1}{1-\rho_t\delta_{t+1}} = \frac{1}{1-\delta_t}$$

It remains only to check that when period t is payment period ($R_t < r$) condition (48) holds. But this follows easily since

$$\frac{a_{2,t+1}}{a_{3,t+1}} = \frac{1}{1-\delta_{t+1}} \geq \frac{1}{1-\rho_t\delta_{t+1}}$$

as $\rho_t < 1$. **QED**

The equilibrium payments Suppose the borrower starts with no debt in period 0 ($D_0 = 0$) and a wealth level W_0. Then from Lemmas 4 - 7 the equilibrium we have constructed is as follows:

In each **payment period** t ($R_t \leq r$) the borrower repays any debt he has (D_t). Note that this implies that if several savings periods follow each other, the only repayment occurs in the first of these.

In each **investment period** t ($R_t > r$) the lender extends a loan of

$$L_t = \frac{W_t\delta_t - D_t}{1-\delta_t}$$

where W_t is the borrower's wealth level, D_t is the borrower's debt level and δ_t is as given by (43). Note that whenever an investment period follows another investment period, then since $D_t = r\left(D_{t-1} + L_{t-1}\right)$, $W_t = R_{t-1}\left(W_{t-1} + L_{t-1}\right)$ and $R_{t-1}\delta_t = r\delta_{t-1}$ we have

$$\begin{aligned} W_t\delta_t - D_t &= r\delta_{t-1}\left(W_{t-1} + L_{t-1}\right) - r\left(D_{t-1} + L_{t-1}\right) \\ &= r\left(W_{t-1}\delta_{t-1} - D_{t-1}\right) - r\left(1-\delta_{t-1}\right)L_{t-1} = 0 \end{aligned}$$

Thus if several investment periods occur consecutively, a loan of

$$\frac{W_t\delta_t}{1-\delta_t} = \frac{W_t\gamma_t}{1-\gamma_t} = L_t^*$$

is granted in the first of these, and no loan is granted in the ones that follow.

In the **penultimate period** $T-1$ the borrower pays of all his debt if he has not already done so.

Since δ_t is larger for smaller t, the **ratio of loan size to wealth** falls as the relationship nears its end.

Bank 1's final profit is 0 (i.e. it exactly breaks even). To find the borrower's utility, note that from Lemma 7

$$a_{2,0} = \frac{a_{3,0}}{1-\delta_0} = \frac{\prod_{s=0}^{T-1} R_s}{1-\delta_0} = \frac{\prod_{s=0}^{T-1} R_s}{1-\gamma_0}$$

Since the borrower's initial debt is $D_0 = 0$, we then have

$$U_0^B(W_0, D_0) = \frac{W_0}{1-\gamma_0} \prod_{s=0}^{T-1} R_s$$

Step 3: Reintroducing the other banks

Finally, we need to show that the equilibrium we have constructed is still an equilibrium when the remaining $M-1$ banks are present prior to date $T-1$.

To see this, suppose to the contrary that there exists a sequence of loans and payments $\left\{\tilde{L}_t^m, \tilde{P}_t^m\right\}$ such that the borrower's final consumption is strictly increased, with $\mathbf{L}_t^1 = (L_t^*, 0, \ldots, 0)$ and $\mathbf{P}_s^1 = (P_s^*, 0, \ldots, 0)$ for all $t \le \tau$ and $s < \tau$ (i.e. τ is the deviation date), and $\left\{\tilde{L}_t^m, \tilde{P}_t^m\right\}$ subgame perfect for dates $t \ge \tau + 1$. Denote the debt levels under the deviation by $\tilde{D}_t^m$.

Since the equilibrium we have constructed achieves the upper bound on the borrower's utility characterized in Proposition 4, bank 1's final profits under these alternative payments must be strictly negative. Thus $\tilde{D}_T^1 < 0$, since the only transfers in period T are from the bank to the borrower.

If $\tilde{D}_{T-1}^m \ge 0$, the debt-default rule $\mathcal{B}_{DD}$ implies that $\tilde{D}_T^m \ge 0$ since the borrower is restricted from depositing funds with any bank $m \ne 1$. But then $\tilde{L}_T^m = 0$, since otherwise bank m would be making negative profits. But then $\tilde{P}_{T-1}^m = 0$ and so $\tilde{L}_{T-1}^m = 0$. Iterating establishes that $\tilde{P}_t^m = \tilde{L}_t^m = 0$ for all $m \ne 1$ and all t.

Similarly, if $\tilde{D}_{T-1}^m < 0$ the debt-default rule $\mathcal{B}_{DD}$ implies that $\tilde{P}_{T-1}^m = 0$ since any positive payment would be seized in entirety. So again we can conclude $\tilde{P}_t^m = \tilde{L}_t^m = 0$.

Thus the deviation must be such that no bank $m \neq 1$ is involved. But in constructing the equilibrium we have already shown that no deviation involving payment just to bank 1 is profitable.

Proof of Proposition 6

Part 1 (Creditor rights): Suppose that contrary to the Proposition's statement there exist values of $\widehat{NP}_0^1$ and $\widehat{NP}_1^1$ such that $\widehat{NP}_1^1 < r\widehat{NP}_0^1$ but with the property that for any $\mu < 1$, there exists a set of payments $\left\{\widehat{NP}_0^m, \widehat{NP}_1^m : m \neq 1\right\}$ such that $\widehat{NP}_0^m = 0$ and $\sum_{m \neq 1} \widehat{NP}_1^m > r\widehat{NP}_0^1 - \widehat{NP}_1^1$ but inequality (14) fails to hold. The proof consists of showing that under these assumptions, there always exists at least some parameter value $x \in \mathcal{X}$ for which the borrower can do strictly better by not repaying bank 1's loan, so that the constrained efficient outcome does not exist as an equilibrium.

Let $\hat{X}$ denote the subset of the parameter space $\mathcal{X}$ for which $L_0^*(x) = \widehat{NP}_0^1$. Suppose $\{NP_0^m, NP_1^m\}$ is an equilibrium that does achieve the constrained efficient outcome. This means that $NP_0^1 = L_0^*\left(\hat{X}\right)$, $NP_0^m = 0$ for $m \neq 1$, and the borrower's final consumption is

$$\hat{U}(x) = \left(W_0 + L_0^*\left(\hat{X}\right)\right) R_0 r - L_0^*\left(\hat{X}\right) r^2$$

and all banks make zero profits.

Suppose for now that the payments $\left\{\widehat{NP}_0^m, \widehat{NP}_1^m : m \neq 1\right\}$ have the property that they completely exhaust the borrower's date 1 wealth, i.e.

$$\sum_{m \neq 1} \widehat{NP}_1^m = \left(W_0 + \widehat{NP}_0^1\right) R_0 - \widehat{NP}_1^1 \tag{49}$$

Choose $\varepsilon > 0$ and $\lambda \in [0,1]$ to be such that the inequality

$$\lambda\left(\left(W_0 + \widehat{NP}_0^1\right) R_0 r - \widehat{NP}_1^1 r\right) - \varepsilon > \left(W_0 + \widehat{NP}_0^1\right) R_0 r - \widehat{NP}_0^1 r^2 = U\left(\hat{X}\right) \tag{50}$$

holds. That is, inequality (50) says that if the borrower can transfer a proportion λ of his date 1 wealth to banks $m \neq 1$ and earn an interest rate r, then he will be strictly better off than repaying bank 1 in full. Note that such choice is always possible, since the left-hand side of (50) is equal to

$$\lambda\left(W_0 + \widehat{NP}_0^1\right) R_0 r - \widehat{NP}_0^1 r^2 + r\left(r\widehat{NP}_0^1 - \lambda\widehat{NP}_1^1\right) - \varepsilon$$

and by supposition $\widehat{NP}_0^1 - \widehat{NP}_1^1 > 0$.

Given inequality (50), it follows that the banks $m \neq 1$ can strictly increase their collective profits to ε by offering to accept saving between dates 1 and 2 at a rate just less than r. Inequality (50) guarantees that the borrower will accept this offer, since doing so yields a utility level strictly greater than $U\left(\hat{X}\right)$. Thus we have established that there cannot be an equilibrium of the type described if (49) holds.

To complete the proof of the lemma it remains only to show that (49) holds for at least some parameter value $x \in \hat{X}$. Consider the line in $\hat{X}$ given by

$$x(\delta) = (W_0, R_0, R_1) = \left(\frac{\delta^2}{r^2 - \delta^2} L_0^*\left(\hat{X}\right), r + \delta, \delta\right) \text{ where } \delta \in (0, r)$$

Since $\widehat{NP}_0^1 = L_0^*\left(\hat{X}\right)$, the borrower's date 1 wealth under the parameter $x(\delta)$ is $\frac{r^2}{r-\delta} L_0^*\left(\hat{X}\right)$. By assumption, $\sum_m \widehat{NP}_1^m > rL_0^*(X)$. So we can always find a value of $\delta \in (0, r)$ such that $\sum_m \widehat{NP}_1^m = \frac{r^2}{r-\delta} L_0^*(X)$, and so (49) holds. This completes the first part of the proof.

Part 2 (Debtor rights): Suppose that contrary to the Proposition's statement there exists an $\hat{L}_0 > 0$ and a $\hat{\mu} < 1$ such that for all $(\mathbf{P}_0, \mathbf{P}_1)$ with $\sum_{m \neq 1} P_1^m > r\left(\hat{L}_0 - P_0^1\right) - P_1^1$ the inequality (15) does not hold. Fix $W_0 = \hat{W}_0$ arbitrarily, and define the set $\hat{X} \subset \mathcal{X}$ to be set of all parameter values x with wealth level $\hat{W}_0$ and such that $L_0^*(x) = \hat{L}_0$. Observe that $(r - R_1) R_0$ is constant over the subset $\hat{X}$.

Note that for the constrained efficient outcome to be an equilibrium at x, bank 1 must use a lending policy with $L_0^1 = \hat{L}_0 = L_0^*\left(\hat{X}\right)$. For any $\varepsilon, \delta > 0$, consider the deviation by bank 1 to a lending policy $\tilde{\mathcal{L}}^1$ with $\tilde{L}_0^1 = \hat{L}_0 - \varepsilon$ and

$$\tilde{l}_2^1(\mathbf{P}_1) = \begin{cases} \left(W_0 + \tilde{L}_0^1\right)(R_1 + \delta) R_0 & \text{if } P_1^1 = \left(W_0 + \tilde{L}_0^1\right) R_0 \\ 0 & \text{otherwise} \end{cases}$$

i.e. at date 1, bank 1 offers to pay a return of $R_1 + \delta$ if the borrower deposit all his wealth. The proof consists of showing that the lending policy $\tilde{\mathcal{L}}^1$ is a profitable deviation for bank 1.

First, assume that given the policy $\tilde{\mathcal{L}}^1$ that the borrower's best response is $\mathbf{P}_0 = \mathbf{0}$ and $\mathbf{P}_1 = \left(\left(W_0 + \tilde{L}_0^1\right) R_0, 0, \ldots, 0\right)$, i.e. the borrower deposits all his date 1 wealth with bank 1. Then at any $x \in \hat{X}$ bank 1 gets

$$-\tilde{L}_0^1 r^2 + (r - R_1 - \delta)\left(W_0 + \tilde{L}_0^1\right) R_0 = \varepsilon\left(r^2 - (r - R_1) R_0\right) - \delta R_0\left(W_0 + \tilde{L}_0^1\right) \quad (51)$$

where we are using the fact that at any $x \in \hat{X}$ we know $-\hat{L}_0 r^2 + (r - R_1)\left(W_0 + \hat{L}_0\right) R_0 = 0$.

It is sufficient to show that the deviation to $\tilde{\mathcal{L}}^1$ is profitable for some $x \in \hat{X}$ (since we require the rule $\mathcal{B}$ to be robust) and some values $\varepsilon, \delta > 0$. We select values of x, ε, δ as follows. First, choose $\mu \geq \hat{\mu}$ such that $(1-\mu) r^2 < (r - R_1) R_0$ for all $x \in \hat{X}$. Choose $\varepsilon \in \left[0, \hat{L}_0\right]$ such that for all $x \in X$

$$(1-\mu) r^2 < \frac{\varepsilon \left(r^2 - (r - R_1) R_0\right)}{W_0 + \hat{L}_0 - \varepsilon} \tag{52}$$

Such a choice is always possible since as $\varepsilon \to L_0$ the RHS tends to $\frac{\hat{L}_0}{W_0}\left(r^2 - (r - R_1) R_0\right) = (r - R_1) R_0$. Let $\hat{x} \in \hat{X}$ be such that

$$\mu r < R_1 \tag{53}$$

(clearly such a choice is always possible by setting R_1 high enough). Finally, choose δ so that

$$(1-\mu) r^2 < \delta R_0 \tag{54}$$

$$\delta R_0 < \varepsilon \frac{\left(r^2 - (r - R_1) R_0\right)}{W_0 + L_0 - \varepsilon} \tag{55}$$

where such a choice is possible by inequality (52).

The right-hand side of (51) is strictly positive given inequality (55), and so bank 1's deviation from $\mathcal{L}^1$ to $\tilde{\mathcal{L}}^1$ is strictly profitable *provided that the borrower responds by depositing all his date* 1 *wealth with bank* 1, i.e., $\mathbf{P}_0 = \mathbf{0}$ and $\mathbf{P}_1 = \left(\left(W_0 + \tilde{L}_0^1\right) R_0, 0, \ldots, 0\right)$. Note that the borrower's final consumption under this choice of $\mathbf{P}_0, \mathbf{P}_1$ given $\tilde{\mathcal{L}}^1$ is

$$\left(W_0 + \tilde{L}_0^1\right) R_0 (R_1 + \delta) = \left(W_0 + \tilde{L}_0^1\right) R_0 R_1 + \left(W_0 + \tilde{L}_0^1\right) R_0 \delta$$

Next, consider any other choice of $\mathbf{P}_0, \mathbf{P}_1$. Necessarily it must feature either $P_0^m > 0$ for some m, and/or $P_1^1 < \left(W_0 + \tilde{L}_0^1\right) R_0$. However, if $P_0^m > 0$ for some m then the payment $P_1^1 = \left(W_0 + \tilde{L}_0^1\right) R_0$ is not feasible,[26] and so either way we have $P_1^1 < \left(W_0 + \tilde{L}_0^1\right) R_0$. The borrower's final consumption is then at most

$$\left(\left(W_0 + \tilde{L}_0^1 - \sum_m P_0^m\right) R_0 - P_1^1\right) R_1 + \sum_{m \neq 1} \left(\beta_1^m r - R_1\right) P_1^m$$

For the case

$$\sum_{m \neq 1} P_1^m > r \left(\tilde{L}_0^1 - P_0^1\right) - P_1^1 \tag{56}$$

[26] This is true provided that no bank $m \neq 1$ offers a lending policy $\mathcal{L}^m$ in which deposits earn a rate of return strictly higher than r. Such a lending policy would generate strictly negative profits. Ruling out bank $m \neq 1$ strategies that yield negative out-of-equilibrium profits is consistent with our assumption that only bank 1 has surplus funds available.

then by supposition the borrower's consumption is less than

$$\left(\left(W_0 + \tilde{L}_0^1 - \sum_m P_0^m\right) R_0 - P_1^1\right) R_1 + (\mu r - R_1) \sum_{m \neq 1} P_1^m$$

By inequality (53) this expression must be strictly less than $\left(W_0 + \tilde{L}_0^1\right) R_0 R_1$. On the other hand, if (56) does not hold then the borrower's consumption is certainly less than

$$r\left(r\left(\tilde{L}_0^1 - P_0^1\right) - P_1^1\right) + R_1\left(W_1 - P_1^1 - \left(r\left(\tilde{L}_0^1 - P_0^1\right) - P_1^1\right)\right)$$

where $W_1 = \left(W_0 + \tilde{L}_0^1 - \sum_m P_0^m\right) R_0$ is the borrower's date 1 wealth. Since

$$\left(W_0 + \tilde{L}_0^1\right) R_0 \geq W_1 \geq r\left(\tilde{L}_0^1 - P_0^1\right) - P_1^1$$

the borrower's consumption under the deviation is less than his consumption from sticking to $\mathbf{P}_0, \mathbf{P}_1$ by at least $(R_1 - r + \delta)\left(r\left(\tilde{L}_0^1 - P_0^1\right) - P_1^1\right)$. Conditions (53) and (54) imply that $R_1 - r + \delta > 0$, again establishing that the borrower will indeed stick to repayments $\mathbf{P}_0, \mathbf{P}_1$. Thus we can conclude that $\mathbf{P}_0 = \mathbf{0}$ and $\mathbf{P}_1 = \left(\left(W_0 + \tilde{L}_0^1\right) R_0, 0, \ldots, 0\right)$ is indeed a strict best response for the borrower, completing the second part of the proof.