Failing to forecast rare events

James Dow (London Business School) and Philip Bond (University of Washington)*

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We analyze the allocation of trading talent across different types of assets, taking into account equilibrium considerations in both labor and financial markets. We identify a strong economic force that leads the highest-skill traders to focus on trading "common event" assets that pay off frequently. Less talented traders instead trade "rare event" assets that pay off only rarely, so that short positions pay off with high probability, i.e., "nickels in front of a steamroller" strategies. This allocation of talent leads to higher bid-ask spreads in common event assets, and reduces the ability of financial markets to predict rare events.

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1 Introduction

One of the main functions performed by the financial sector is to forecast future events. However, many observers have expressed concern that, as they perceive it, the majority of forecasting activity is devoted to forecasting frequent but relatively unimportant events. The financial system has been criticized for its failure to predict the financial crisis of 2007-08.[1] Taleb (2007) asks "[w]hy do we keep focusing on the minutiae, not the possible significant large events, in spite of the obvious evidence of their huge influence?" Relatedly, many commentators have criticized the "quarterly earnings cycle" and the amount of effort devoted to forecasting firms' next earnings announcements (see, e.g., Kay (2012)).[2] Relatedly also, there are concerns that the risk-management departments of financial institutions—which in principle are concerned with predicting and mitigating large but infrequent events—have trouble recruiting and retaining high-quality employees (e.g., Palm, 2014).[3]

This paper analyzes the economic incentives for forecasting events of different frequencies. Are there systematic economic forces that push people to focus on predicting everyday events as opposed to rare events? Specifically, since trading is the main way that agents profit from information in financial markets, are there forces that favor trading securities whose payoffs depend on frequent events? Do traders of different skills trade different kinds of securities? Does the aggregate amount of trading skill dedicated to predicting rare and frequent events differ? And are rare events more or less likely to be predicted as a result?

By analyzing a simple equilibrium model of the financial sector, we identify a strong economic force that leads individuals with more skill to trade the common event asset, while individuals with less skill trade the rare event asset. Moreover, because this leads to more informed trading in the common event asset, the bid-ask spread for this asset is higher; and the ability of financial markets to predict rare events is reduced.

[1] *Financial Times*, November 25 2008, "The economic forecasters' failing vision."
[2] *Financial Times*, February 29, 2012, "Investors should ignore the rustles in the undergrowth."
[3] *American Banker*, September 9, 2014, "Why Banks Face a Risk Management Talent Shortage."

The key feature of our model is that it combines equilibrium analysis of the financial market (using a standard Glosten and Milgrom (1985) model of bid and ask prices) with equilibrium analysis of the labor market (using a standard Roy (1951) model). Specifically, individuals choose between the two "occupations"of trading a binary-payoff asset in which both states are reasonably likely—a "common event" asset—and trading an alternative "rare event" asset in which one state is overwhelmingly more likely than the other state. Traders are subject to position limits, on which we impose only minimal assumptions.

To convey the intuition for our results, it is useful to first consider a benchmark in which we consider only the equilibrium conditions of the labor market, without imposing equilibrium in the financial market. Specifically, we consider the benchmark case in which financial assets trade at prices equal to their unconditional expected payoffs.[4] Under a natural assumption about position limits (see subsection 3.1), the expected return to skill is equal in the common- and rare-event assets. However, the trading patterns in the two assets are very different: in the common event market, the skilled trader often takes a moderately sized position, while in the rare event market the trader occasionally buys a large position but usually holds a small short position.[5]

With this benchmark in mind, consider how trading profits change when financial asset prices are determined in equilibrium. Our simple yet central observation is that the rare-event asset must have a non-neglible bid-ask spread. This is because if the bid-ask spread were instead very small, trading the rare-event asset would be profitable even for low-skilled traders. But then substantial trading skill would be devoted to the rare-event asset, leading to a significant bid-ask spread.

The non-negligible bid-ask spread on the rare-event asset, combined with position limits, means that traders are unable to adopt very large positions in the rare-event asset. This implies that prediction skill has low value when it is devoted to the rare event. Traders will

[4]These prices violate the equilibrium conditions of the financial market, because they allow informed traders to make large profits at others' expense.

[5]Note that a short position in the rare event asset is a bet that the event will not occur.

only very rarely predict that the asset will pay off, and so will very rarely want to adopt a long position. When they do so, the size of their long position cannot be extremely large, because of position limits and the non-negligible bid-ask spread. Most of the time traders will predict that the rare-event asset will not pay off, and accordingly will adopt a short position (e.g., the carry trade, selling out-of-the-money puts). But this short position is not very profitable, since the bid price of the asset is very low.

Next we consider financial markets'—as opposed to individuals'—ability to predict future rare events. The distribution of talent across different assets is a contributing factor. The highest skilled agents choose to specialize in predicting the frequent event, while the rare event asset is traded by only relatively unskilled traders. This means that, in general, aggregate trading activity contains more information about common events than about rare events, relative to a counterfactual benchmark in which the people trading rare and common assets are exogenously interchanged. Our formal results exhibit sufficient conditions for this implication.

Our prediction on the allocation of skill matches informal perceptions that a lot of forecasting "talent" is devoted to forecasting frequent events. It is also consistent with the view that many standard trading strategies such as the carry trade, selling out-of-the-money put options, etc., are "nickels in front of steamroller strategies" that are carried out by people with mediocre talents. These are short positions on rare events, which our model predicts will only attract low-skill traders.

In addition to predictions on the allocation of skill to different types of assets, our model delivers predictions for variation in bid-ask spreads across different assets. The bid-ask spread predictions are easiest to apply to bonds (or corresponding CDS positions), which have a payoff structure well-approximated by the binary payoff assumption of our model.[6] The rare event asset in our model corresponds to a short position in a high-rated bond, while the common event asset corresponds to a short position in a low-rated bond. Consequently,

[6]In the case of zero recovery after default, the payoff structure of a bond exactly matches the binary payoff assumption.

our model predicts that low-rated bonds have larger bid-ask spreads than high-rated bonds. This is consistent with evidence from both sovereign and corporate bond markets (see Calice et al (2013) for sovereign and Edwards, Harris, and Piwowar (2007), Goldstein and Hotchkiss (2011) and Benmelech and Bergman (2017) for corporate).

Since Hirshleifer (1971), if not much longer, economists have been aware that the social value of information generally diverges from the private incentive to produce information.[7] There are many reasons why this can happen, and for that reason our paper is not focused on welfare economics. However there is an implicit welfare conclusion to be drawn from our analysis, that the best talent may not be employed in studying unlikely events even if they are economically important.

The existing literature on information acquisition has primarily dealt with how investors who are ex ante homogeneous divide their information acquisition efforts across different assets. In contrast, we study the matching between hetorogeneous investors, who differ in terms of skill, and heterogeneous assets, which differ in terms of payoff frequency. In other words, we study the inter-personal division of labor in information acquisition, while the existing literature focuses on the intra-personal allocation of information acquisition.

Van Niewerburgh and Veldkamp (2010) analyze an investor's choice of which assets to acquire information about before deciding portfolio holdings. Acquiring information about an asset helps the investor solve the problem of optimal portfolio choice; such an asset effectively becomes less risky. They establish conditions under which the investor specializes and acquires information about just one asset. An important difference between their paper and ours is that in our analysis asset prices, including the bid-ask spread, are determined endogenously.

Veldkamp (2006) analyzes a model in which traders buy information from information providers, and information production enjoys economies of scale. The model is along the lines of Grossman and Stiglitz (1980) but with multiple assets; the motive for acquiring infor-

[7]For instance, predicting an earnings announcement a few days in advance can be profitable but socially useless, while an important invention may not enrich its inventor.

mation is to make trading profits at the expense of less informed agents, and to make better portfolio allocation decisions (e.g. risk-return tradeoffs). She shows that, in equilibrium, different traders choose to observe the same signals, thereby increasing the comovement of asset prices.

In Peng and Xiong (2006) the representative investor has a cognitive constraint (which could also be interpreted as a cost of information production) which leads to choosing signals that are informative about many assets. The benefit from information is to improve the consumption-savings decision, unlike our paper where traders benefit from information because it helps them make money at the expense of uninformed traders.

Agents will collect similar signals if there is strategic complementarity in information production, and the literature has identified a number of reasons why this could arise. In Dow and Gorton (1994) traders hope to enter positions in an inefficent maket and close them out in an efficient market, there is only one asset, but a trader's profits from receiving a signal about the asset are higher if other (later) traders receive a signal because it means they are able to profitably close out their positions at a price without a waiting for the cash flow to arrive. In Dow and Gorton (1997), Boot and Thakor (1997) and Dow, Goldstein and Guembel (2017) information is also a strategic complement because having many agents collect information makes the price more informative, which increases the information sensitivity of the asset, which in turn inctreases the value of information. None of these papers analyse which kinds of assets traders should specialize in.

2 Model

2.1 Assets

There are two financial assets which we call the r-asset and the c-asset ("rare" and "common"). Each asset pays either 0 or 1 (that is, the price should be understood as the price per unit of payoff in the event the asset pays off). We model the assets as associated with two independent random variables, ψ^r and ψ^c, each distributed uniformly over $[0,1]$. The r-asset pays 1 if $\psi^r \leq q^r$ for some constant q^r and 0 otherwise. Likewise, the c-asset pays 1 if $\psi^c \leq q^c$ for a constant q^c. So the probability that the j-asset pays off is $\Pr\left(\psi^j \leq q^j\right) = q^j$. We will focus on the case in which q^r approaches zero. There is a period in which the assets trade, after which payoffs are realized. A trader who takes a long position in the j-asset trades at a price P_L^j (the ask price). Likewise, if he takes a short position, he trades at a price P_S^j (the bid price). Subsection 2.5 below details how prices are determined.

2.2 Skilled traders

There is a continuum of risk-neutral skilled traders.[8] Each trader observes either an informative signal or a purely noisy signal. When a trader observes a signal $s^j \in [0,1]$ no-one, including the trader, knows whether the signal is informative or not. However, there is publicly observable heterogeneity in the likelihood that a trader will observe an informative signal: each trader has a publicly observed probability α of receiving informative signals. We refer to a trader's α as his "skill." The population distribution of α is given by measure $\bar{\mu}$; we assume the distribution admits a density, which we denote by g.

Collecting information takes time. To capture this, we assume that signals have an opportunity cost: each trader must choose between receiving signals about ψ^r or signals about ψ^c. A trader with skill α who chooses to observe a signal about ψ^j observes the true realization with probability α, and otherwise observes the realization of a noise term uniformly distributed over $[0,1]$. This assumption has the natural property that the unconditional probability distribution of signals is the same for all α.

After observing his signal, a trader chooses whether to trade. He can take either long or short positions. Let $V^j(\alpha)$ denote the expected payoff of a skilled trader with skill α who specializes in the j-asset. Moreover, we define $V^0(\alpha) \equiv 0$ for the payoff of agents who trade neither asset.

[8] Alternatively, a trader may be risk-averse, but be insured by a risk-neutral employer.

We assume traders live for just one period. We extend this to two periods in an online appendix, where we consider learning about skill.

2.3 Position limits

Traders face position limits, corresponding to margin constraints and limits on how much they can borrow to finance their positions. Position limits enter our analysis because a potential attraction of trading the r-asset is that its price is low, and so a trader can adopt large positions; position limits determine how large this position can be. For most of our analysis, including all our main results, we allow for a very broad class of position limits, potentially depending on asset prices, and impose only weak assumptions.

To motivate these weak assumptions, consider the specific example in which position limits are determined by the requirement that traders must have sufficient cash ("margin") to close out their positions in all states. This is a natural assumption when the asset payoff is a binary random variable as we have assumed, or more generally has a finite support. In other words, for a long position, the lowest realization of asset value must be sufficient to pay off any loan that was taken out to buy the asset, in addition to the trader's initial wealth. For a short position, the initial wealth plus proceeds from the short sale should be enough to buy back the borrowed stock at its highest possible value. In this case, the largest long position is W/P_L^j and the largest short position is $W/(1-P_S^j)$, for initial wealth W.[9]

In general, the largest feasible long and short positions are $h_L^j\left(P_L^j\right)$ and $h_S^j\left(P_S^j\right)$, respectively, where the functions h_L^j and h_S^j satisfy the following pair of mild assumptions (both of which are satisfied by the above example):

Assumption 1 *h_L^j and h_S^j are continuous functions over $(0,\infty)$ and take strictly positive values.*

[9]The short position limit arises as follows. If a trader short sells x units, he has total wealth $W + xP_S^j$, which is sufficient collateral for $W + xP_S^j$ short positions. So the largest feasible short position is given by the solution to $x = W + xP_S^j$.

Assumption 2 $\lim_{P \to 0} P h_S^j(P) = 0$.

Although Assumption 1 is weak, it is actually slightly stronger than necessary. Specifically, our analysis requires just one of h_B^j and h_S^j to be strictly positive. In particular, our results are unchanged if short positions are impossible.

Assumption 2 ensures that short positions do not grow too fast as the rare event becomes rarer and its price (presumably) falls. This is a very weak assumption in the sense that one would expect the short position limit to *decrease* in the price, since a lower price corresponds to fewer short proceeds to collateralize future obligations. Note that there is no need for an analogous assumption on long position limits $h_L^j(P)$: the reason is that as we show below, ask prices P_L^j remain bounded away from zero in equilibrium (Lemma 2).

2.4 Liquidity traders

In addition to skilled traders, there is a continuum of uninformed traders who trade for non-informational reasons. We refer to these traders as "liquidity traders." Each liquidity trader receives an endowment shock that gives him a strong desire for resources in a particular state. A measure λ^r of liquidity traders are r-liquidity traders, and each receives a shock $\chi^r \sim U[0,1]$, meaning that he wants resources in state $\psi^r = \chi^r$. Similarly, a measure λ^c of liquidity traders are c-liquidity traders, and each receives a shock $\chi^c \sim U[0,1]$, meaning that he wants resources in state $\psi^c = \chi^c$. Except in Section 6, we make no assumption on whether and how liquidity shocks are correlated across liquidity traders.

For simplicity, we assume that j-liquidity trader preferences for resources in state χ^j are lexicographic, so that each j-liquidity trader takes as large a long position as possible in the j-asset as possible if $\chi^j \leq q^j$, and as large a short position as possible if $\chi^j > q^j$. The long and short position limits for j-liquidity traders are the same as for skilled traders, namely h_L^j and h_S^j. Given this, a j-liquidity trader buys $h_L^j(P_L^j)$ units of the j-asset if he experiences shock $\chi^j \leq q^j$, and short sells $h_S^j(P_S^j)$ units of the j-asset if he experiences shock $\chi^j > q^j$.

Consequently, the expected number of buy orders for the j-asset from liquidity traders equals $q^j \lambda^j h_L^j(P_L^j)$, while the expected number of sell orders is $(1-q^j)\, \lambda^j h_S^j(P_S^j)$. The assumption that the expected number of liquidity buy orders approaches zero as q^r approaches zero is economically reasonable given our motivation of uninformed trades as resulting from a liquidity need, but it is not innocuous, and is important for the results.

Liquidity traders are rational in the sense of making choices that are consistent with a complete, transitive preference ordering, although their demands for the asset are price inelastic. Allowing price elastic demands for the liquidity traders would make the derivation of equilibrium more complex. We have motivated non-informational trades as liquidity trades. Two alternative motivations are that these trades are simply "noise" trades, or alternatively, they come from traders who have zero skill but mistakenly believe themselves to have high skill (Dow and Gorton, 2008). Our analysis below is independent of which of the three motivations is assumed, so long as the volume of non-informational long positions is proportional to the probability of the rare event.

2.5 Financial market structure

Traders face different bid and ask prices P_L^j and P_S^j. Bid and ask prices are set by the zero profit condition of a competitive market maker. The interpretation is that there are many market makers each posting binding quotes for bid and ask prices. Traders arrive simultaneously and can fulfil their orders at these prices. Each market maker takes into account the equilibrium skill and behavior of skilled and liquidity traders when posting prices:

$$\begin{aligned} E\left[\text{buys}|\psi^j \le q^j\right] \Pr\left(\psi^j \le q^j\right)\left(P_L^j - 1\right) + E\left[\text{buys}|\psi^j > q^j\right] \Pr\left(\psi^j > q^j\right) P_L^j &= 0 \\ E\left[\text{sales}|\psi^j \le q^j\right] \Pr\left(\psi^j \le q^j\right)\left(1 - P_S^j\right) + E\left[\text{sales}|\psi^j > q^j\right] \Pr\left(\psi^j > q^j\right)\left(-P_S^j\right) &= 0. \end{aligned}$$

Rearranging and simplifying gives

$$P_L^j = q^j \frac{E\left[\text{buys}|\psi^j \leq q^j\right]}{E\left[\text{buys}\right]} \tag{1}$$

$$P_S^j = q^j \frac{E\left[\text{sales}|\psi^j \leq q^j\right]}{E\left[\text{sales}\right]}. \tag{2}$$

This price setting mechanism is similar to that in Glosten and Milgrom (1985).[10]

2.6 Minimum skill levels for profitable trading

Consider a skilled trader of skill α who specializes in the j-asset. If he observes a signal $s^j \in [0,1]$ and takes a long position at the ask price P_L^j, his expected profits on each unit bought are

$$\Pr\left(\psi^j \leq q^j | s^j\right) - P_L^j, \tag{3}$$

while if he takes a short position at the bid price P_S^j, his expected profits on each unit sold short are

$$P_S^j - \Pr\left(\psi^j \leq q^j | s^j\right). \tag{4}$$

Evaluating,

$$\Pr\left(\psi^j \leq q^j | s^j\right) = \alpha \mathbf{1}_{s^j \leq q^j} + (1-\alpha)\Pr\left(\psi^j \leq q^j\right) = \alpha \mathbf{1}_{s^j \leq q^j} + (1-\alpha) q^j, \tag{5}$$

since the asset pays off either if the signal is informative (probability α) and indicates the asset will valuable ($s^j \leq q^j$); or, if it is uninformative, with the unconditional payoff probability q^j. Consequently, a skilled trader buys[11] after seeing signal $s^j \leq q^j$ if and only if his skill α exceeds

$$\frac{P_L^j - q^j}{1 - q^j}, \tag{6}$$

[10] Glosten and Milgrom (1985) also contains results on how prices evolve as new orders are processed.

[11] It is straightforward to verify that if $P_S^j \leq q^r$ then a skilled trader would never sell after observing $s^j \leq q^j$. Similarly, if $P_L^j \geq q^r$ then a skilled trader would never buy after observing $s^j > q^j$. We verify below that $P_L^j \geq q^j \geq P_S^j$ indeed holds in equilibrium.

Likewise, a skilled trader sells after seeing signal $s^j > q^j$ if and only if his skill exceeds

$$1 - \frac{P_S^j}{q^j}. \tag{7}$$

3 Equilibrium in financial and labour markets

3.1 Benchmark: No financial market equilibrium

We start by considering, as a benchmark, how traders would allocate themselves if, contrary to equilibrium conditions, assets were priced at expected value with no bid-ask spread instead of reflecting the amount of informed trading in that asset. Suppose that traders in the j-asset can buy or sell as much as they want at the unconditional expected value:

$$P_L^j = P_S^j = q^j. \tag{8}$$

With probability $1-\alpha$, a trader of skill α has an uninformative signal and makes zero profits. With probability α, he gets an informative signal and makes money. For this benchmark we wish to calculate profits, so we require a specific functional form for the position limits. As noted previously, a natural specification of the position limits is $h_L^j(P) = \frac{W}{P}$ and $h_S^j(P) = \frac{W}{1-P}$ (the largest positions that can be supported by initial wealth W). In this case, the expected profits of a trader in the j-asset who receives an informative signal are

$$q^j \frac{W}{P_L^j}(1 - P_L^j) + (1 - q^j)\frac{W}{1 - P_S^j}P_S^j = q^j \frac{W}{q^j}(1 - q^j) + (1 - q^j)\frac{W}{1 - q^j}q^j = W.$$

Hence a trader of skill α specializing in the j-asset makes expected profits αW, which is the same for both assets. So in this benchmark, with a very natural specification of position limits, skilled traders are indifferent between trading the two assets regardless of their skill.

Notice that this conclusion depends on traders being able to take both long and short positions. As the asset becomes rarer, long positions become more profitable while short positions become less profitable. If they are restricted to long positions only, the expected payoff to specializing in the j-asset is

$$\alpha q^j \frac{W}{P_L^j}(1 - P_L^j) = \alpha q^j \frac{W}{q^j}(1 - q^j) = \alpha W(1 - q^j)$$

so they prefer the rare asset. This is in line with the view that rare event assets are attractive because they are so cheap. Investors can take very big positions in them although, as we will argue below, this argument is fallacious because it fails to recognize that equilibrium prices respond to the level of informed trading activity.

3.2 Equilibrium in both financial and labor markets

Clearly, the assumption in (8) that traders can both buy and sell the asset at its unconditional expected value is flawed in any economy with a positive measure of skilled traders. With skilled traders present, assets do not trade at their unconditional expected value; they trade at prices that reflect the incidence of informed trading.

In contrast, our goal is to jointly characterize traders' choices of which assets to trade and the prices of those assets. Equilibrium in financial markets requires that prices reflect the level of informed trade. Equilibrium in labor markets requires that traders make optimal choices about which asset to specialize in, given financial asset prices.

Given our previous discussion of the minimum skill required to trade in light of the bid-ask spread, some agents will have a skill level that is too low to trade in either asset, so they will choose to do nothing. Others will specialize in trading the r-asset or the c-asset.

Definition 1 *An equilibrium consists of prices* $(P_L^r, P_S^r, P_L^c, P_S^c)$ *and an allocation of skilled traders* (μ^r, μ^c, μ^0) *across the r-asset, the c-asset and doing nothing, such that:*

1. Labor market equilibrium:

(a) Optimal choice of asset: For almost all skill levels α *and for all* $i \in \{r, c, 0\}$ *such that* $\mu^i(\alpha) > 0, V^i(\alpha) \geq V^j(\alpha)$ *for all* $j \in \{r, c, 0\}$.

(b) Labour markets clear: $\mu^r(\alpha) + \mu^c(\alpha) + \mu^0(\alpha) = \bar{\mu}(\alpha)$ for almost all skill levels α.

2. Financial market equilibrium: Given profit-maximizing trading by skilled traders, prices satisfy (1) and (2).

Proposition 1 below establishes equilibrium existence by standard continuity arguments.

4 Prices conditional on skill allocation

In this section we solve for the financial market equilibrium given the allocation of skill. Given a labour market allocation (μ^r, μ^c, μ^0), write A^j for the aggregate skill in asset j, i.e.,

$$A^j \equiv \int \alpha \mu^j (d\alpha), \tag{9}$$

and N^j for the mass ("number") of skilled traders in asset j, i.e.,

$$N^j \equiv \int \mu^j (d\alpha). \tag{10}$$

Define

$$X^j \equiv \frac{A^j}{\lambda^j + N^j}. \tag{11}$$

Intuitively, a market maker who fills a buy or sell order is concerned about the informational advantage of the counterparty, which in our setting amounts to the probability the order comes from a skilled trader as opposed to a liquidity trader, multiplied by the expected amount of skill given that the trader is skilled. This is

$$\frac{N^j}{\lambda^j + N^j} \cdot \frac{A^j}{N^j} = \frac{A^j}{\lambda^j + N^j} = X^j,$$

so the bid and ask prices for the asset should reflect X^j. In addition, a trader who considers specializing in an asset will also care about X^j; if it is too high in relation to his skill we would expect that it may not be profitable to enter that market. The intuition (familiar from

Glosten and Milgrom (1985) and the microstructure literature) is that the bid-ask spread is a measure of the amount of skilled trading. But the more skilled trading there is, the larger the bid-ask spread, and the harder it is for low-skill traders to make profits, hence the higher the threshold level of skill required to trade profitably. So we should expect X^j to be related to both the bid-ask spread and the minimum skill required to profitably trade the asset. This intuition can be made precise:

Lemma 1 *Given (A^r, N^r, A^c, N^c), prices for assets $j = r, c$ are:*

$$P_L^j = q^j + \left(1 - q^j\right) X^j \tag{12}$$

$$P_S^j = q^j - q^j X^j, \tag{13}$$

and the minimum skill required both to profitably buy the j-asset after observing signal $s^j \leq q^j$ and to profitably sell the j-asset after observing signal $s^j > q^j$ is X^j.

It is immediate from (12) and (13) that the bid-ask spread is

$$P_L^j - P_S^j = X^j. \tag{14}$$

Note that $A^j \leq N^j$, so

$$X^j \in \left[0, \frac{1}{1+\lambda^j}\right]. \tag{15}$$

5 Equilibrium analysis

5.1 Equilibrium existence

To establish equilibrium existence, we construct a correspondence from bid-ask spreads (X^r, X^c) into themselves: first, given bid-ask spreads, we use the labor market equilibrium condition to determine which asset a trader with skill α specializes in, and second, given the allocation of traders across assets, we use the financial market equilibrium conditions (12) and (13) to determine the bid-ask spread. Since the bid-ask spread X^j is continuous in the aggregate skill measures A^j and N^j, Kakutani's fixed point theorem implies that the correspondence described has a fixed point, at which both labor and financial markets are in equilibrium.

Proposition 1 *An equilibrium exists.*

The remainder of this section characterizes equilibrium properties.

5.2 The bid-ask spread in the r-asset is bounded away from zero

We start by showing that the combination of equilibrium in financial and labor markets implies that both the bid-ask spread (X^r) in the r-asset, and the minimum skill level required to trade it (also X^r), are bounded away from 0, even as the r-event grows very rare ($q^r \to 0$). Although relatively simple, this result is central to our analysis.

To build intuition, suppose there is just one asset in the economy, and the probability that it pays off approaches zero. One might conjecture that the ask price of this asset would also approach zero, because that is the expected value of the payoff. For example a fixed percentage markup over the expected value would imply that the price converges to zero in the limit. Then, all agents, however low their chance of receiving an informative signal about the asset payoff, would start to trade and buy the asset when they received a buy signal, i.e., $s^r \leq q^r$. But given a positive measure of skilled traders buy the asset after observing $s^r \leq q^r$, the ask price is informative and cannot be close to zero. This is a contradiction, so in the limit the ask price would have to be bounded away from zero. Hence, a zero ask price in the limit would violate a very basic equilibrium condition.

More constructively, we can see what will happen in the limit: as the payoff probability approaches zero, the price approaches a limit that is higher than zero. At this price, higher-skilled traders trade while lower-skilled traders do not trade. In between, there is a marginal type of trader whose skill is just high enough to be indifferent between trading and not trading. Given this, the ask price is higher than the expected value by a premium that reflects the average informativeness of signals of all types that are higher than this marginal type. Informally, this premium reflects the cumulative "brainpower" of traders who buy when they receive a positive signal. In equilibrium, the premium in turn implies that the marginal type is indeed indifferent between buying and not buying.

We have explained the intuition in terms of an economy where the rare event asset is the only asset, but the reasoning in the economy where there is also a common asset is similar. Lemma 2 formalizes this argument, and accounts for the fact skilled traders choose between the r-asset and c-asset:

Lemma 2 *Both the bid-ask spread for the r-asset and the minimum skill level required to trade the r-asset remain bounded away from 0 as $q^r \to 0$, i.e., there exists $\underline{x}$ such that $X^r \geq \underline{x}$ for all q^r small.*

An immediate but important consequence is:

Corollary 1 *The ask price P_L^r is bounded away from 0 as the unconditional expected value of the r-asset q^r approaches 0.*

Moreover:

Corollary 2 *Aggregate skill in the r-asset, A^r, is bounded away from 0 even as q^r approaches 0.*

Lemma 2 relies on the probability of liquidity buys approaching zero as the probability of the rare event approaches zero. Our microfoundation of liquidity trades, based on hedging needs, gives one possible economic argument for this assumption. More generally, this property should hold in a very general class of economies: it is hard to justify why uninformed "noise" or liquidity traders would remain interested in taking a long position in an asset that pays off vanishingly rarely.

5.3 Skill allocation across assets

A skilled trader (specialized in the j-asset) observes a buy signal $s^j \leq q^j$ with probability q^j, and a sell signal $s^j > q^j$ with probability $1 - q^j$. By Lemma 1, combined with (3), (4) and (5), the expected payoff of a skilled trader with skill $\alpha \geq X^j$ who specializes in the j-asset is

$$q^j h_L^j \left(P_L^j\right) \left(\alpha + (1-\alpha) q^j - P_L^j\right) + \left(1 - q^j\right) h_S^j \left(P_S^j\right) \left(P_S^j - (1-\alpha) q^j\right). \tag{16}$$

The first term corresponds to long positions, and the second term to short positions. In the first term, q^j is the probability of taking a long position, $h_L^j \left(P_L^j\right)$ is the size of the position, and the profit on each unit of the asset is the expected payoff (which is the probability the payoff equals 1) minus the price paid. In the second term, $(1 - q^j)$ is the probability of taking a short position, $h_S^j \left(P_S^j\right)$ is the size of the position, and the profit on each unit of the position is the price received minus the expected payoff (which is the probability that the payoff equals 1). Substituting in for the bid and ask prices using (12) and (13), this payoff can be expressed in terms of X^j. Combined with that fact that a trader always has the option of not trading, the expected payoff is

$$V^j(\alpha) = \max\{0, q^j \left(1 - q^j\right) \left(h_L^j \left(q^j + \left(1 - q^j\right) X^j\right) + h_S^j \left(q^j - q^j X^j\right)\right) \left(\alpha - X^j\right)\}. \tag{17}$$

The value given in this expression is the expected payoff for a trader who can trade both long and short positions. By setting $h_S^j = 0$ or $h_L^j = 0$ we can see the value if a trader can take only long positions or only short positions. The latter case is of interest because a short position in the rare event asset is equivalent to a long position in an asset that nearly always pays off a small return, but occasionally loses all the capital. Our main results go through for both these cases.

We now consider the marginal value of an extra increment of skill in trading an asset.

From (17), for $\alpha > X^j$,

$$\frac{\partial V^j(\alpha)}{\partial \alpha} = q^j \left(1 - q^j\right) \left(h_L^j \left(q^j + \left(1 - q^j\right) X^j\right) + h_S^j \left(q^j - q^j X^j\right)\right). \tag{18}$$

From this expression, and using Corollary 1 and Assumption 2, the marginal value of skill is very low in the r-asset because q^j is low. Formally:

Lemma 3 *As $q^r \to 0$, the marginal value of skill in the r-asset (18) approaches* 0.

To understand Lemma 3, notice from (17) that for a skilled trader who chooses to trade, profits as a function of α are a straight line. The slope of this line is the marginal value of skill. Therefore, to show the marginal value of skill goes to zero as $q^r \to 0$, we can show that trading profits go to zero. There are two economic effects underlying this. First, as $q^r \to 0$, traders only rarely buy the r-asset. Consequently, the expected profit from long positions also becomes small unless traders are able to make enormous profits from long positions—which could only happen if they took enormous long positions, as they do in the benchmark model of Section 3 without financial market equilibrium. But by Corollary 1, the dual requirement of equilibrium in financial and labor markets means that the ask price of the r-asset stays bounded away from 0. The lower bound on the price implies an upper bound on the size of the positions, so traders' long positions cannot grow arbitrarily large, implying that the expected profit from long positions indeed approaches 0.

Second, turning to short positions, as $q^r \to 0$ traders specializing in the r-asset nearly always adopt short positions. A trader with skill α has an expected profit on each short position of $P_S^j - (1-\alpha) q^j = q^j (\alpha - X^j)$, which converges to 0 as $q^r \to 0$. It would only be possible for a trader to make non-negligible expected profits on the short position if he could take a large enough short position, but Assumption 2 stops the short position from growing large (as noted above, it is natural for position limits on short positions to *de*crease as price falls, so this is a very weak assumption).

In contrast, the marginal value of skill in common asset does not go to zero (this is

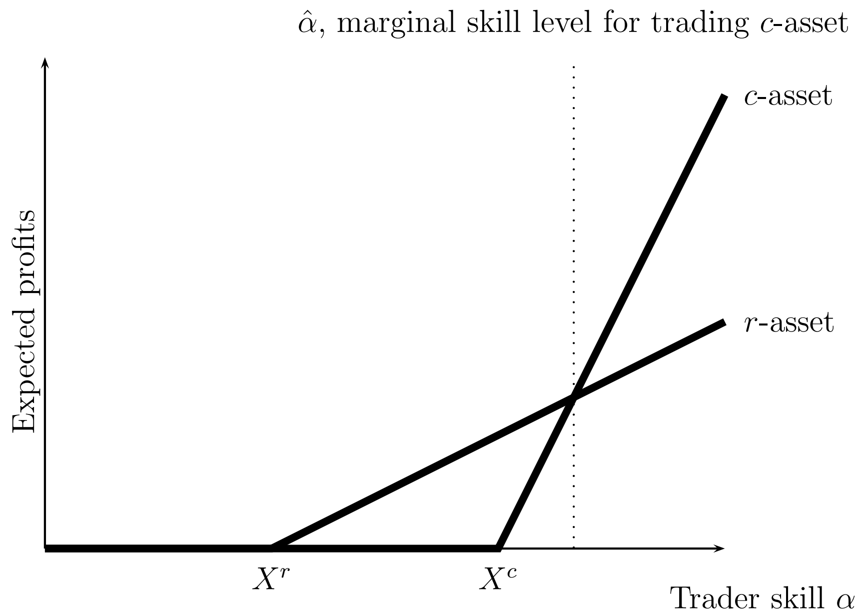

Figure 1: Equilibrium allocation of traders

shown in the proof of Proposition 2). The higher marginal value of skill in the common asset implies that high skill workers have a comparative advantage in the common asset and hence specialize in that asset. High skill workers are better at trading both assets, i.e., they have an absolute advantage compared to low skill workers. But an additional unit of skill is more valuable in the common asset, giving higher skilled agents a comparative advantage in that asset. Hence in equilibrium there is a threshold skill level so that agents with skill below the threshold choose the r-asset while those with skill above the threshold choose the c-asset.

Proposition 2 *For all q^r sufficiently small, the minimum skill required to profitably trade the r-asset is below the minimum skill to profitably trade the c-asset, i.e., $X^r < X^c$. Moreover, there exists $\hat{\alpha} > X^c$ such that traders with skill $\alpha \in (X^r, \hat{\alpha})$ trade the r-asset and traders with skill $\alpha > \hat{\alpha}$ trade the c-asset.*

Proposition 2 is illustrated by Figure 1, which shows how expected profits from specializing in each asset depend on the trader's skill level.

Proposition 2 predicts that (among active traders) the least-skilled traders specialize in the r-asset. As noted, most of the time, they take a short position in this asset. The short position nets a small immediate profit, but exposes the trader to a small risk of a much larger loss in the future if the rare event is realized. Hence, our model predicts that the least skilled traders pursue what are often described as "picking-up nickels in front of a steamroller" strategies, such as the carry trade in currency markets, or writing out-of-the-money puts.

Because traders in the r-asset are relatively unskilled, only a few them manage to successfully predict the rare event when it actually occurs. Hence our model rationalizes the fact that rare events are foreseen by few people, even though the payoff to successfully forecasting such events might seem very large. Nonetheless, a few traders do successfully predict the rare event. As we show in an online appendix, the posterior estimate of these traders' skill is very high.

5.4 Bid-ask spreads

As discussed in the introduction, in addition to our model's implication that skill is concentrated on forecasting common rather than rare events (Proposition 2), our analysis delivers the following prediction for bid-ask spreads:

Corollary 3 *For all q^r sufficiently small, the bid-ask price is smaller for the r-asset than the c-asset, $P_L^r - P_S^r < P_L^c - P_S^c$.*

Corollary 3 follows immediately from Lemma 1 and Proposition 2.

Again as discussed in the introduction, this prediction is easiest to interpret for bonds, where our binary-payoff assumption is a good approximation to reality. The r-asset corresponds to a short position in a highly-rated bond, so that the payoff state corresponds to bond default (a rare event for highly-rated bonds). Similarly, the c-asset corresponds to a short position in a lower-rated bond, where the payoff state of bond default is more likely. So Corollary 3 predicts that the bid-ask spread for bonds is lower for highly-rated bonds. This is consistent with empirical evidence from both corporate and sovereign debt markets (see references in the introduction). Closely related, one should also see wider bid-ask spreads on CDS contracts for which the reference asset carries more default risk.

5.5 Absolute versus proportional bid-ask spreads

The above discussion concerns absolute bid-ask spreads. It is important to note, however, that our analysis implies that while the r-asset will have a smaller absolute bid-ask spread than the c-asset, its absolute bid-ask spread is bounded away from zero so it will have a very large proportional bid-ask spread.

Specifically, we calculate the proportional bid-ask spread as the ratio between the absolute bid-ask spread, $P_L^r - P_S^r$, and the mid-point quote, $\frac{1}{2}\left(P_L^j + P_S^j\right)$. Note that the proportional bid-ask spread has a maximum possible value of 2 (corresponding to $P_S^j = 0$).

Corollary 4 *The proportional bid-ask spread* $\frac{P_L^r - P_S^r}{\frac{1}{2}\left(P_L^r + P_S^r\right)}$ *of the* r*-asset approaches the maximum value of* 2 *as* q^r *approaches* 0*.*

Corollary 4 is an immediate consequence of Lemma 2, which states that X^r remains bounded away from 0 even as q^r approaches 0, coupled with the basic fact (see Lemma 1) that the bid price P_S^r is bounded above by the unconditional expected payoff q^r.

Economically, Corollary 4 follows from the fact that although, in equilibrium, less skill is devoted to the r-asset than to the c-asset, some skill is nonetheless devoted to the r-asset—and in particular, the amount of skill devoted to the r-asset remains bounded away from 0.

Very closely related to Corollary 4 is:

Corollary 5 *The unconditional expected gross return from a long position in the* r*-asset,* $\frac{q^r}{P_L^r}$*, approaches* 0 *as* q^r *approaches* 0*.*

Hence the expected return for buying assets with a very small chance of payoff is very low. This is consistent with the low returns to buying out-of-money puts and calls in option markets (the "smile" in implied volatilities), and with the low returns to wagers on extreme underdogs in betting markets (the "longshot-favorite bias").

6 Predictions from the market

Thus far, we have focused on the ability of individual traders to forecast rare events. In this section, we instead consider the information content of aggregate trading activity.

In our setting, bid and ask prices arise as in Glosten and Milgrom (1985), and as such, are independent of the true state and hence uninformative. In contrast, the aggregate order flow is informative. Accordingly, we consider what an outside observer who observes the total numbers of buy and sell orders for asset j can infer about the likelihood of the j-event. ($\psi^j \leq q^j$). To aid interpretation, note that the total numbers of buy and sell orders can, alternatively, be inferred from seeing a combination of any two of: (i) the average transaction price (in addition to posted bid and ask prices), (ii) aggregate volume, and (iii) order flow imbalance.

Write L^j and S^j for total buy (long) and sell (short) orders for asset j. Write λ_L^j and λ_S^j for the mass of liquidity traders who buy and sell asset j. Write N_L^j and N_S^j for the mass of skilled traders who buy and sell asset j. Hence

$$\begin{aligned} L^j &= \left(\lambda_L^j + N_L^j\right) h_L^j \left(P_L^j\right) \\ S^j &= \left(\lambda_S^j + N_S^j\right) h_S^j \left(P_S^j\right). \end{aligned}$$

Recall that both liquidity traders and active skilled traders always trade in one direction or the other. Consequently, $\lambda_L^j + \lambda_S^j = \lambda^j$ and $N_L^j + N_S^j = N^j$. Hence observing the total number of buy and sell orders (L^j, S^j) has the same information content as simply observing the total number of buy orders, L^j.

The information content of the aggregate order flow depends critically on the correlation among liquidity traders, and similarly, on the correlation among skilled traders. For example, if liquidity trades are uncorrelated, and if skilled trades are uncorrelated conditional on the realization of ψ^j (a natural assumption), then by the law of large numbers L^j will perfectly reveal whether or not $\psi^j \leq q^j$. In the literature, it is assumed liquidity trades are correlated so as to prevent full revelation (e.g., Grossman and Stiglitz 1980, Hellwig 1980, Kyle 1985). In this section, we assume they are perfectly correlated for simplicity while skilled trades are uncorrelated conditional on ψ^j (see, e.g., Grossman 1976, Hellwig 1980), so that

$$N_L^j = A^j \mathbf{1}_{\psi^j \leq q^j} + \left(N^j - A^j\right) q^j. \tag{19}$$

(We obtain similar results if we allow for correlation among skilled trades; notes are available from the authors upon request.)

Given (19), the information content of the aggregate order flows in asset j is the same as the information content of

$$\tilde{L}^j \equiv \frac{L^j}{h_L^j\left(P_L^j\right)} - \left(N^j - A^j\right) q^j = A^j \mathbf{1}_{\psi^j \leq q^j} + \lambda_L^j. \tag{20}$$

From (20), one can see that the aggregate skill A^j deployed to asset j is the key factor in determining the information content of the aggregate order flow. Informally this corresponds to adding up the IQ of the traders in each asset.

So far we have shown (Proposition 2) that all traders in the r-asset have skill below a certain threshold $\hat{\alpha}$ while all traders in the c-asset have skill higher than that threshold. Among other things, this implies that the average skill of people trading the r-asset is lower than that of people trading the c-asset, i.e., $\frac{A^r}{N^r} < \frac{A^c}{N^c}$, and relatedly, that the bid-ask spread is smaller for the r-asset than for the c-asset, i.e., $\frac{A^r}{\lambda^r+N^r} < \frac{A^c}{\lambda^c+N^c}$. We now investigate whether aggregate skill is likewise lower, i.e., $A^r < A^c$.

6.1 Lower aggregate skill in the r-asset

Clearly, a sufficient condition for aggregate skill devoted to the r-asset to be lower is that fewer people trade it, $N^r \leq N^c$. More generally, aggregate skill devoted to the r-asset is lower provided that N^r does not exceed N^c by too much. But if there are a very large number of low-skill traders in the r-asset, and not many high-skill traders in the c-asset, it appears that aggregate skill in the r-asset could be higher.

Intuitively, N^r can only exceed N^c by a large amount if the density function g of the skill distribution declines rapidly in skill α. Our next result formalizes this intuition, and gives a simple sufficient condition on the slope of the density function g that guarantees that N^r is not too large relative to N^c, and hence in turn that less aggregate skill is indeed deployed to the r-asset than to the c-asset. Let $\overline{\alpha}$ denote the maximum of the support of g.

Proposition 3 *If there are equal numbers of liquidity traders in the two assets,*

$$\lambda^r = \lambda^c \tag{21}$$

and the density of skill g satisfies

$$x \int_z^x \alpha g(\alpha)\, d\alpha > z^2 g(z)(x - z) \ \text{ for all } z < x \leq \overline{\alpha}, \tag{22}$$

then for any q^r sufficiently small, less aggregate skill is deployed to the r-asset, i.e., $A^r < A^c$.

Condition (22) of Proposition 3 holds trivially if the density function is weakly increasing in skill. In particular, condition (22) holds if skill α is distributed uniformly over any subinterval of $[0, 1]$. A class of distributions for which condition (22) is violated is the set of triangular distributions defined by $g(\alpha) = \frac{2}{\overline{\alpha}^2}(\overline{\alpha} - \alpha)$. But even for this case, less aggregate skill is deployed to the r-asset unless the mass of liquidity traders is extremely small. Specifically (and for any value of maximal skill $\overline{\alpha}$), it can be shown that there is less aggregate skill in the r-asset so long as liquidity traders are more than 0.8% of skilled traders

($\lambda^r = \lambda^c > 0.008$, where the total mass of skilled traders is normalized to 1).

More generally, the conclusion of Proposition 3 holds for *any* distribution of skill provided that the mass of liquidity traders is sufficiently large.[12]

6.2 Market predictions from the r-asset are less informative

Our main result of this section uses results from the theory of information orderings (see Blackwell 1953, Lehmann 1988). It requires the mild assumption that the density of λ_L^r is log-concave. Recall that, as discussed in subsection 6.1, the condition $A^c > A^r$ is typically satisfied in equilibrium. There is more aggregate IQ deployed in the c-asset. We can use this to compare the accuracy of learning in the two assets. We consider the impact of exogenously interchanging the sets of investors trading the two asset types, i.e., A^r trade the c-asset while A^c trade the r-asset. We show that this switch increases the informativeness of the aggregate order flow in the r-asset.

To say that one information structure is more Blackwell-informative than another is a strong statement. It means that any agent who needs to take any decision would prefer to have the former information structure. It is only a partial ordering of information structures. However in this case the event agents are trying to predict (the asset pays off) is binary, which as Jewitt (2007) observes, simplifies the application of Blackwell's theorem.

Proposition 4 *Suppose the density of λ_L^r is log-concave. If there are equal numbers of liquidity traders in the two assets ((21) holds), and $A^c > A^r$, then the aggregate order flow of the r-asset would be more Blackwell informative if the sets of people trading the r-asset and c-asset were exogenously switched.*

By exogenously switching the sets of people who trade the r-asset and c-asset, we mean

[12] In brief, the argument is as follows. As in Proposition 3, we assume that $\lambda^c = \lambda^r$. As $\lambda^c = \lambda^r \to \infty$, it is straightforward to show that $N^c + N^r$ is bounded away from 0. (Intuitively, if there are many liquidity traders then it is easy for skilled traders to profitably trade.) If $N^r \to 0$ but $N^c \not\to 0$, it is immediate that $A^r < A^c$. If instead $N^r \not\to 0$, then $\frac{X^r}{X^c}$ is bounded away from 1 (from above). We know $\frac{A^r}{A^c} = \frac{X^r}{X^c}\frac{\lambda^c+N^c}{\lambda^r+N^r}$. Since N^c and N^r are both bounded, we know $\frac{\lambda^c+N^c}{\lambda^r+N^r} \to 1$ as $\lambda^c = \lambda^r \to \infty$. It follows that $\frac{A^r}{A^c} < 1$ for $\lambda^c = \lambda^r$ large enough.

that everyone who used to trade the c-asset (i.e., with skill α exceeding the threshold level $\hat{\alpha}$) is now restricted to either trading the r-asset or doing nothing, and similarly, that everyone who used to trade the r-asset (skill $\alpha \in [X_R, \hat{\alpha}]$) is now restricted to either trading the c-asset or doing nothing. The option of doing nothing potentially matters because after the people trading the two assets are switched, asset prices change, and consequently it is possible that not everyone who previously traded the c-asset wants to trade the r-asset at its new equilibrium prices. The role of condition (21) is to ensure that profitably trading the r-asset is not much more difficult than trading the c-asset solely because of a lack of liquidity traders; if instead λ^r were much lower than λ^c, it is possible that many traders who used to trade the c-asset drop out of trading after they are exogenously switched to the r-asset.

Proposition 4 implies a welfare statement if the expected social value of predicting the rare event is sufficiently large compared to the social value of predicting common event. In particular, if the rare event is especially important, information is under-produced.

7 Conclusion

One of the main functions performed by the financial sector is to forecast future events. However, many observers have expressed concern that, as they perceive it, the majority of forecasting activity is devoted to forecasting frequent but relatively unimportant events. In this paper we analyze a simple equilibrium model of the number and skill of financial sector participants who are allocated to predict different types of events. The key feature of our model is that it combines equilibrium analysis of the financial market (using a standard Glosten and Milgrom (1985) model of bid and ask prices) with equilibrium analysis of the labor market (using a standard Roy (1951) model).

Our main result is that this simple equilibrium model delivers the following strong prediction: Individuals with more skill trade the common event asset, while individuals with less skill trade the rare event asset. Moreover, because this leads to more informed trading in the common event asset, the bid-ask spread for this asset is higher. In other words, there is more information produced about the frequent event.

Our prediction on the allocation of skill matches perceptions that a lot of forecasting "talent" is devoted to forecasting frequent events. It is also consistent with the view that many standard trading strategies (e.g., the carry trade, selling out-of-the-money put options, etc.) are "nickels in front of steamroller strategies" that are carried out by people with mediocre talents.

The bid-ask spread predictions are easiest to apply to bonds (or corresponding CDS positions), which have a payoff structure well-approximated by the binary payoff assumption of our model. Our model predicts that low-rated bonds have larger bid-ask spreads than high-rated bonds. This is consistent with evidence from both sovereign and corporate bond markets (see Calice et al (2013) for sovereign and Edwards, Harris, and Piwowar (2007), Goldstein and Hotchkiss (2011) and Benmelech and Bergman (2017) for corporate).

Finally, we show that the endogenous distribution of talent across different types of assets reduces financial markets'—as opposed to individuals'—ability to predict future rare events. Specifically, we show that financial markets generally produce less information about rare events relative to information production in a counterfactual benchmark in which the people trading rare and common assets are exogenously interchanged.

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Appendix: Proofs of results stated in main text

Proof of Lemma 1: We first compute prices under the conjecture that any skilled trader who trades j-asset takes both long and short positions; and then confirm this conjecture. Under this conjecture:

$$\begin{aligned}
E\left[\text{buys}|\psi^j\right] &= q^j\lambda^j h_L^j\left(P_L^j\right) + \int \left(\alpha\mathbf{1}_{\psi^j\leq q^j} + (1-\alpha)\,q^j\right)\mu^j\,(d\alpha)\,h_L^j\left(P_L^j\right) \\
&= \left(q^j\lambda^j + A^j\left(\mathbf{1}_{\psi^j\leq q^j} - q^j\right) + q^jN^j\right)h_L^j\left(P_L^j\right) \\
E\left[\text{sells}|\psi^j\right] &= \left(1-q^j\right)\lambda^j h_S^j\left(P_L^j\right) + \int \left(\alpha\mathbf{1}_{\psi^j> q^j} + (1-\alpha)\left(1-q^j\right)\right)\mu^j\,(d\alpha)\,h_S^j\left(P_S^j\right). \\
&= \left(\left(1-q^j\right)\lambda^j + A^j\left(\mathbf{1}_{\psi^j> q^j} - \left(1-q^j\right)\right) + \left(1-q^j\right)N^j\right)h_S^j\left(P_S^j\right).
\end{aligned}$$

Hence from (1) and (2),

$$P_L^j = q^j\frac{q^j\lambda^j + A^j\left(1-q^j\right) + q^jN^j}{q^j\lambda^j + q^jN^j} = q^j\left(1+\frac{A^j}{\lambda^j+N^j}\frac{1-q^j}{q^j}\right) \tag{23}$$

$$P_S^j = q^j\frac{\left(1-q^j\right)\lambda^j - A^j\left(1-q^j\right) + \left(1-q^j\right)N^j}{\left(1-q\right)^j\lambda^j + \left(1-q^j\right)N^j} = q^j\left(1-\frac{A^j}{\lambda^j+N^j}\right). \tag{24}$$

Substituting for X^j in expressions (23) and (24) yields prices (12) and (13).

Given prices (12) and (13), from (6) the minimum skill level required to profitably buy the

j-asset after observing signal $s^j \le q^j$ is $\frac{q^j+(1-q^j)X^j-q^j}{1-q^j} = X^j$, while from (7) the minimum skill level required to profitably sell the j-asset after observing signal $s^j > q^j$ is $\frac{q^j-(q^j-q^jX^j)}{q^j} = X^j$. Hence a skilled trader trades j-asset if and only if he takes both long and short positions. **QED**

Proof of Proposition 1: We construct a correspondence $\xi : [0,1]^2 \to [0,1]^2$ as follows. For any $(X^r, X^c) \in [0,1]^2$, construct bid and ask prices according to (12) and (13). Given prices, allocate skilled traders to the asset where their expected profit is higher. For the case of indifference, allow for all randomizations between the two assets. Finally, given the allocation of skilled traders, evaluate new values of (X^r, X^c) according to (11). The resulting correspondence is upper-hemicontinuous, and closed- and compact valued. By Kakutani's fixed point theorem it has a fixed point, which corresponds to an equilibrium of the economy. **QED**

Proof of Lemma 2: Suppose to the contrary that there exists some sequence $\{q^r\}$ such that $q^r \to 0$ and the associated $X^r \to 0$.

First, consider the case in which X^c stays bounded away from 0, by $\underline{x}^c$ say. But then $X^r \to 0$ implies that skilled traders in the skill interval $[X^r, \underline{x}^c]$ certainly trade the rare asset. But then from (11), $X^r \not\to 0$, a contradiction.

Second, consider the case in which $X^c \to 0$ for some subsequence. So all skilled traders trade something. But by (11), this contradicts $X^c + X^r \to 0$, completing the proof. **QED**

Proof of Corollary 2: From Lemma 2, there exists $\underline{x} > 0$ such that $X^r \ge \underline{x}$ even as as $q^r \to 0$. Since $A^r \le N^r$, it follows that $\frac{A^r}{\lambda^r+A^r} \ge \frac{A^r}{\lambda^r+N^r} \ge \underline{x}$, and hence that there exists $\underline{A}$ such that $A^r \ge \underline{A}$ even as as $q^r \to 0$. **QED**

Proof of Lemma 3: By Lemma 2, as $q^r \to 0$, the term $q^r(1-q^r)\,h^r_L(q^r + (1-q^r)X^r)$ in equation (18) approaches 0. The remaining term $q^r(1-q^r)\,h^r_S(q^r - q^rX^r)$ can be written $\frac{1-q^r}{1-X^r}q^r(1-X^r)\,h^r_S(q^r(1-X^r))$, of which $q^r(1-X^r)\,h^r_S(q^r(1-X^r))$ approaches 0 as $q^r \to 0$ by Assumption 2 while $\frac{1-q^r}{1-X^r}$ is bounded above (from (15)). **QED**

Proof of Proposition 2: Note first that, for all q^r, by (18) the marginal value of skill in the c-asset is bounded below by

$$q^c (1 - q^c) \min_{\tilde{X} \in \left[0, \frac{1}{1+\lambda^c}\right]} h_L^c \left(q^c + (1 - q^c) \tilde{X} \right) > 0.$$

In contrast, from Lemma 3 we know the marginal value of skill in the r-asset approaches 0.

To establish $X^r < X^c$ when q^r is small, suppose to the contrary that $X^r \geq X^c$ even as q^r grows small. From the above comparison of the marginal value of skill, it follows that no-one trades the r-asset for q^r sufficiently small (since the payoff functions are linear and upward sloping, and the payoff for the r-asset has a larger intercept and a smaller slope). But then $X^r = 0$, which contradicts Lemma 2 and so establishes that $X^r < X^c$.

Given $X^r < X^c$ and the comparison of the marginal value of skill, the existence of a cutoff skill level $\hat{\alpha}$ is immediate.

Finally, $\hat{\alpha} > X^c$ because at $\alpha = X^c$, the r-asset has a strictly positive payoff while the c-asset has a zero payoff. **QED**

Proof of Proposition 3: Write λ for the common value of λ^r and λ^c. For any $x \in (0, \overline{\alpha}]$, define $f(x)$ as the unique solution in $(0, x)$ to

$$f(x) \left(\lambda + \int_{f(x)}^{x} g(\alpha)\, d\alpha \right) - \int_{f(x)}^{x} \alpha g(\alpha)\, d\alpha = 0.$$

The existence of $f(x)$ follows from the fact that

$$z \left(\lambda + \int_{z}^{x} g(\alpha)\, d\alpha \right) - \int_{z}^{x} \alpha g(\alpha)\, d\alpha$$

is strictly negative at $z = 0$ and strictly positive at $z = x$. Uniqueness follows from the fact that differentiation implies that this same function is strictly increasing in z. Moreover, and for use below, note that

$$f'(x)\left(\lambda+\int_{f(x)}^{x} g(\alpha)\,d\alpha\right)+f(x)\,g(x)-f(x)\,f'(x)\,g(f(x))-xg(x)+f'(x)\,f(x)\,g(f(x))=0,$$

and hence

$$f'(x)=\frac{g(x)\,(x-f(x))}{\lambda+\int_{f(x)}^{x} g(\alpha)\,d\alpha}=\frac{f(x)\,g(x)\,(x-f(x))}{\int_{f(x)}^{x}\alpha g(\alpha)\,d\alpha}.$$

Define

$$\begin{aligned}\bar{X}^c &= f(\overline{\alpha})\\ \bar{X}^r &= f\left(\bar{X}^c\right),\end{aligned}$$

so that

$$\begin{aligned}\bar{X}^c &= \frac{\int_{\bar{X}^c}^{\overline{\alpha}}\alpha g(\alpha)\,d\alpha}{\lambda+\int_{\bar{X}^c}^{\overline{\alpha}} g(\alpha)\,d\alpha}\\ \bar{X}^r &= \frac{\int_{\bar{X}^r}^{\bar{X}^c}\alpha g(\alpha)\,d\alpha}{\lambda+\int_{\bar{X}^r}^{\bar{X}^c} g(\alpha)\,d\alpha}.\end{aligned}$$

From the observations about the marginal value of skill in the proof of Proposition 2, we know that as $q^r \to 0$, $X^c \to \bar{X}^c$ and $X^r \to \bar{X}^r$. So to establish the result, we show

$$\int_{\bar{X}^c}^{\overline{\alpha}}\alpha g(\alpha)\,d\alpha>\int_{\bar{X}^r}^{\bar{X}^c}\alpha g(\alpha)\,d\alpha,$$

or equivalently,

$$\int_{f(\overline{\alpha})}^{\overline{\alpha}}\alpha g(\alpha)\,d\alpha>\int_{f\left(\bar{X}^c\right)}^{\bar{X}^c}\alpha g(\alpha)\,d\alpha.$$

Since $\overline{\alpha}>f(\overline{\alpha})=\bar{X}^c$, it suffices to show that $\int_{f(x)}^{x}\alpha g(\alpha)\,d\alpha$ is increasing in x, or equivalently,

$$xg(x)-f'(x)\,f(x)\,g(f(x))>0,$$

which substituting in the earlier expression for $f'(x)$ is equivalent to

$$xg(x) > \frac{f(x)\,g(x)\,(x - f(x))}{\int_{f(x)}^{x} \alpha g(\alpha)\,d\alpha} f(x)\,g(f(x)),$$

i.e.,

$$x \int_{f(x)}^{x} \alpha g(\alpha)\,d\alpha > f(x)^2\,g(f(x))\,(x - f(x)).$$

This inequality is implied by (22), completing the proof. **QED**

Proof of Proposition 4: First note that when traders who trade the c-asset in equilibrium are exogenously reallocated to trading the r-asset, they are happy to actively trade the r-asset. This follows because (by (21)), the minimum skill required to profitably trade the r-asset after the exogenous switch coincides with the minimum skill required to profitably trade the c-asset before the switch.

The result then follows from the following claim:

Claim: The Blackwell informativeness of the aggregate order flow in the r-asset is increasing in the aggregate skill A of the people actively trading the r-asset.

Proof of claim: Let ω_0^r and ω_1^r respectively denote the events that the r-asset does not pay off, $\psi^r > q^r$ and that it does pay off, $\psi^r \leq q^r$. Let H denote the distribution function of λ_L^r. As discussed in the main text, if aggregate skill A actively trades the r-asset, then the information content of the aggregate order flow of the r-asset with respect to $\omega \in \{\omega_0^r, \omega_1^r\}$ is the same as the information content of $A\mathbf{1}_{\omega=\omega_1^r} + \lambda_L^r$. Let $F(\cdot;\omega, A)$ denote the distribution function of $A\mathbf{1}_{\omega=\omega_1^r} + \lambda_L^r$.

Evaluating, $F(y;\omega,A) = H\left(y - A\mathbf{1}_{\omega=\omega_1^r}\right)$ and $F^{-1}(t;\omega,A) = H^{-1}(t) + A\mathbf{1}_{\omega=\omega_1^r}$.

Consider any pair of aggregate skill levels A and $\tilde{A} > A$. Hence

$$F^{-1}\left(F(y;\omega,A);\omega,\tilde{A}\right) = H^{-1}\left(H\left(y - A\mathbf{1}_{\omega=\omega_1^r}\right)\right) + \tilde{A}\mathbf{1}_{\omega=\omega_1^r} = y + \left(\tilde{A} - A\right)\mathbf{1}_{\omega=\omega_1^r}.$$

Consequently, for any y,

$$F^{-1}\left(F\left(y;\omega_1^r,A\right);\omega_1^r,\tilde{A}\right)\geq F^{-1}\left(F\left(y;\omega_0^r,A\right);\omega_0^r,\tilde{A}\right),$$

i.e., the r-asset order flow is more informative in the Lehmann sense (Lehmann 1988) if it is actively traded by a set of people with aggregrate skill $\tilde{A}$ rather than A. Since the density of λ_L^r is log-concave, the distribution function $F\left(y;\omega,A\right)$ has the monotone likelihood ratio property. Since $\{\omega_0^r,\omega_1^r\}$ is a binary set, it follows from Proposition 1 in Jewitt (2007) that an increase in aggregate skill A makes the r-asset order flow more informative in the Blackwell sense (Blackwell 1953), completing the proof of the claim, and hence the proof. **QED**

A Online Appendix: Career concerns

Trading on rare and common events may also differ because of learning effects. Agents may be unsure about their underlying ability to predict trading outcomes, and the dynamics of learning may be different for rare and common events.[13] Although this is not the main focus of our paper, we consider a variant of our model in which agents work for more than one period (specifically, two periods), allowing for updating about skill levels.

To summarize the analysis of this appendix:

Successful prediction of a rare event is a very strong indicator of skill, and leads to a very favorable posterior on skill. Consequently the people with the highest perceived skill in the economy trade are to be found trading the rare event asset. As such, traders who have successfully executed successful long trades in the r-asset will be held in very high regard—as is indeed the case, for example, for people who correctly predicted the collapse in house prices in 2007-2008.

Nonetheless, because the rare event is unusual, learning about skill plays only a very small role in the market for the rare event asset, in a sense that we formalize below. Moreover, learning about skill is less important for the rare event asset than for the common event asset in the following sense: in equilibrium, an agent who is indifferent between specializing in each of the two assets and then chooses the rare event asset will trade in the first period, but if he instead chooses the common event asset he will trade only after a positive updating of skill following a prior successful prediction.

A.1 Dynamic model

To analyze career concerns, we need a model of traders who operate in multiple periods. To keep the analysis as transparent as possible, we assume that each skilled trader has a career

[13] There is a literature that explores agents' incentives to take risks because of the contracts they have signed with principals, or because of career concerns (Trueman (1988), Dow and Gorton (1997)). The predictions of such models can be quite sensitive to the assumed functional forms of contracts or of the contracting environment. These considerations are beyond the scope of this paper. In this paper agents do not distort their decisions in order to manipulate other agents.

lasting two periods. The decision of which type of asset to specialize in over the trader's entire career is made prior to the first period, and affects both periods. In each period, a trader has the option of making a prediction without trading. This could be interpreted as making only a very small investment. To keep the economy stationary, we assume that each period a new generation of skilled traders enters the economy, with skill distributed according to the measure $\frac{\bar{\mu}}{2}$.

Also to ensure stationarity, we assume there are a continuum of r-assets, and a continuum of c-assets, and traders who specialize in r-assets (respectively, c-assets) are randomly allocated across the continuum of different r-assets (respectively, c-assets) at the start of each period. Thus, each market we study has both young and old traders, and while the old traders have previous experience, the outcomes of that experience are independent.

Career concerns only arise in our framework if predictive ability (skill) is persistent. To keep the exposition as transparent as possible, we focus on the case in which predictive ability is as persistent as possible: specifically, if a trader receives an informative signal in one period, he does so in all periods. In addition to aiding exposition, this assumption also makes the learning channel as strong as possible. As noted, our results below show that learning plays a limited role; as such, reducing the degree of persistence would only strengthen the results.

To avoid confusion as agents live for two periods, an agent of skill α should be interpreted to mean "an agent whose prior probability in the first period of receiving an informative signal is α." This means that in the first period, the agent will receive an informative signal with probability α and an uninformative signal with probability $1 - \alpha$. He knows this probability. When the agent trades in the second period, he is still "an agent of skill α" but because he will receive an informative signal if and only if he previously received an informative signal, his probability of receiving an informative signal is updated by Bayes' rule. The updating is different for the r-asset and the c-event.

A.2 Learning

To aid exposition, we denote by ω_1^j the event $\psi^j \leq q^j$ in which asset j pays off, and similarly, ω_0^j the is event $\psi^j > q^j$ in which asset j pays 0. For conciseness, we assume $q^c \leq \frac{1}{2}$ but the other case is also straightforward to handle. The unconditional probabilities that a trader with skill α successfully predicts states ω_1^j and ω_0^j respectively are denoted $p_1^j(\alpha)$ and $p_0^j(\alpha)$:

$$\begin{aligned}
p_1^j(\alpha) &= q^j\left((1-\alpha)q^j+\alpha\right) = q^j\left(q^j+\alpha\left(1-q^j\right)\right) \\
p_0^j(\alpha) &= \left(1-q^j\right)\left((1-\alpha)\left(1-q^j\right)+\alpha\right) = \left(1-q^j\right)\left(1-q^j+\alpha q^j\right).
\end{aligned}$$

Consequently, the posteriors that a trader receives an informative signal—henceforth, simply "posterior skill"—given successful prediction of ω_1^j and ω_0^j are respectively

$$\begin{aligned}
\alpha_1^{j\prime}(\alpha) &= \frac{\alpha q^j}{p_1^j(\alpha)} = \frac{\alpha}{q^j+\alpha\left(1-q^j\right)}. \\
\alpha_0^{j\prime}(\alpha) &= \frac{\alpha\left(1-q^j\right)}{p_0^j(\alpha)} = \frac{\alpha}{1-q^j+\alpha q^j}.
\end{aligned}$$

Notice that as $q^j \to 0$, the posterior $\alpha_1^{j\prime}(\alpha)$ approaches 1: a trader who has successfully predicted an extremely unlikely event has almost certainly done so because of skill, not by chance. This formalizes our observation above that traders who have successfully executed successful long trades in the r-asset will be held in very high regard.

Our assumption that skill is completely persistent means that the posterior skill level of a trader who failed to correctly predict the state is 0, since this failure reveals the trader did not receive an informative signal. This assumption considerably simplifies the exposition, but does not qualitatively affect our results.

Below, we make use of the inverses of the updating functions $\alpha_1^{j\prime}$ and $\alpha_0^{j\prime}$:

$$\left(\alpha_1^{j\prime}\right)^{-1}(\alpha) = \frac{\alpha q^j}{1-\alpha\left(1-q^j\right)} \tag{A-1}$$

$$\left(\alpha_0^{j\prime}\right)^{-1}(\alpha) = \frac{\alpha\left(1-q^j\right)}{1-\alpha q^j}. \tag{A-2}$$

Note that $\alpha_1^{j\prime}(\alpha) \geq \alpha_0^{j\prime}(\alpha) \geq \alpha$ and $\left(\alpha_1^{j\prime}\right)^{-1}(\alpha) \leq \left(\alpha_0^{j\prime}\right)^{-1}(\alpha) \leq \alpha$, as $q^j \leq \frac{1}{2}$. In other words, successful prediction that the event ω_1^j occurs leads to more updating than successful prediction that ω_0^j does not occur.

A.3 Expected profits from specializing in the j-asset

A skilled trader's decision to specialize in one asset over another reflects expected lifetime trading profits in each asset. From (17), expected lifetime trading profits in the j-asset are

$$\begin{aligned} V^j(\alpha) &= q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right) \\ &\times \left(\max\left\{0,\alpha-X^j\right\}+p_1^j(\alpha)\max\left\{0,\alpha_1^{j\prime}(\alpha)-X^j\right\}+p_0^j(\alpha)\max\left\{0,\alpha_0^{j\prime}(\alpha)-X^j\right\}\right). \end{aligned}$$

Here, the last two terms in parentheses correspond to an experienced trader's profits after, respectively, successfully predicting ω_1^j and ω_0^j.

When traders live two periods, the payoff function V^j is convex, and piecewise linear with three kinks, at $\left(\alpha_1^{j\prime}\right)^{-1}(X^j)$, then $\left(\alpha_0^{j\prime}\right)^{-1}(X^j)$, and then X^j. Economically, for skill levels below $\left(\alpha_1^{j\prime}\right)^{-1}(X^j)$, even the posterior assessment of skill after successful prediction of ω_1^j is too low to justify trading. For skill levels in $\left(\left(\alpha_1^{j\prime}\right)^{-1}(X^j), \left(\alpha_0^{j\prime}\right)^{-1}(X^j)\right)$ a trader trades only after successful prediction of ω_1^j when young. For skill levels in $\left(\left(\alpha_0^{j\prime}\right)^{-1}(X^j), X^j\right)$ a trader trades after both successful prediction of ω_1^j and ω_0^j when young, but not after unsuccessful prediction. Finally, for skill levels above X^j, a trader traders when young, and continues trading when old provided he made profits (i.e., predicted successfully) when young.

A.4 Basic equilibrium analysis

We start by reproducing several results from the one-period economy. First, and as in Lemma 2, the minimum skill required to profitably trade the r-asset is bounded away from zero. The basic economic force is the same as before. The only new elements in the proof are associated with the need to handle updating about skill levels.

Lemma A-1 *There exists some* $\underline{x} > 0$ *such that* $X^r \geq \underline{x}$ *for all* q^r *small.*

As before, an immediate but important corollary of Lemma A-1 is that the ask price remains bounded away from the fair price q^j.

Next, we reproduce Lemma 3 from the one-period economy: even taking the value of learning into account, the marginal value of skill in the r-asset still approaches 0.

Lemma A-2 *As* $q^r \to 0$*, the marginal value of skill in the* r*-asset approaches* 0*.*

Given Lemma A-2, similar arguments as in the one-period economy imply that it is the lowest skill traders who actively trade who specialize in the r-asset (see Proposition 2). As with other results, the only new elements in the proof are those associated with learning about a trader's skill:

Proposition A-1 *For all* q^r *small enough,* $0 < (\alpha_1^{r\prime})^{-1}(X^r) < (\alpha_1^{c\prime})^{-1}(X^c)$*.*

Finally, the bid-ask spread is larger for the c-asset, just as in the one-period economy:

Proposition A-2 *For all* q^r *small enough, the bid-ask spread in the c-asset is larger,* $X^c > X^r$*.*

A.5 Career concerns

With the above results in hand, we next analyze the role of learning.

First, although Proposition A-1 shows that the lowest skill traders who actively trade specialize in the r-asset, it is nonetheless the case that the traders in the economy with the *highest* identified skill trade the r-asset. This is a consequence of the power of updating from successfully predicting event ω_1^r:

Corollary A-1 *As* $q^r \to 0$*, the posterior skill of some people who trade the* r*-asset approaches* 1*, i.e., there exists some* α *who trades the* r*-asset such that* $\alpha_1^{r\prime}(\alpha) \to 1$*.*

Somewhat anecdotally, this is consistent with the descriptions in Lewis (2011), in which the fund managers who predicted the housing crisis were unheralded prior to the crisis, but attracted large fund inflows after the crisis.

At the same time, in spite of this powerful updating, in the aggregate there is only limited updating from successful prediction of the rare event, in the following sense. For $j = r, c$, define A^{Lj} as the aggregate skill trading asset j that previously successfully predicted the event that asset j paid off. Define μ_y^j as the measure of young traders who trade asset j. Then

$$A^{Lj} = \int p_1^j(\alpha)\, \alpha_1^{j\prime}(\alpha)\, \mu_y^j(d\alpha)\,. \tag{A-3}$$

Proposition A-3 *Learning from successful prediction of ω_1^r plays a very small role in total trade in the r-asset: $A^{Lr}/A^r \to 0$ as $q^r \to 0$.*

In other words, most traders who trade the r-asset are either young traders, or experienced traders who successfully predicted the non-occurrence of a rare event (ω_0^r) when young.

From Proposition A-1, it is the least skilled traders who trade the r-asset. Just as in the one-period economy, there is a marginal skill level at which traders are indifferent between specializing in the two assets. A distinct sense in which learning plays only a limited role in the r-asset is that this marginal trader behaves very differently in each of these alternative tracks. If he specializes in the c-asset, he trades only after he has first made a successful prediction. The reason is that the bid-ask spread is relatively high in the c-asset (see Proposition A-2), and so only relatively skilled traders can profitably trade this asset. In contrast, if he specializes in the r-asset, he trades the asset right from the very start of his career, because the bid-ask spread is lower, and hence less skill is required for profitable trading.

Proposition A-4 *For q^r sufficiently small, there is a skill level $\hat{\alpha} \in (X^r, X^c)$ such that all traders with initial skill in $\left[(\alpha_1^{r\prime})^{-1}(X^r), \hat{\alpha}\right)$ specialize in the r-asset, and all traders with* *initial skill above $\hat{\alpha}$ specialize in the c-asset. The marginal-skill trader $\hat{\alpha}$ faces a choice between: trading the r-asset immediately, and trading the c-asset only in the second period, after successful prediction in the first period.*

A.6 Proofs for results on career concerns

Proof of Lemma A-1: Suppose to the contrary that there exists some sequence $\{q^r\}$ such that $q^r \to 0$ and the associated $X^r \to 0$.

First, consider the case in which $\left(\alpha_1^{c\prime}\right)^{-1}(X^c)$ stays bounded away from 0, by $\underline{\alpha}$ say. But then $X^r \to 0$ implies that traders will skill in the interval $[X^r, \underline{\alpha}]$ trade the r-asset when young. But then $X^r \nrightarrow 0$, contradicting the original supposition.

Second, consider the case in which $\left(\alpha_1^{c\prime}\right)^{-1}(X^c) \to 0$ for some subsequence. From (A-1), $X^c \to 0$. Hence the total skill A^c in the c-asset must approach 0. Likewise, by supposition, $X^r \to 0$, so the total skill A^r in the r-asset must approach 0. But the combination of these two statements is impossible, since because both X^c and $X^r \to 0$, the fraction of skilled traders that trades at least one of the two assets when young approaches 1. The contradiction completes the proof. **QED**

Proof of Lemma A-2: By the same argument as in the proof of Lemma 3, the two-period profit function $V^r(1) \to 0$, even when evaluated at the maximum skill level $\alpha = 1$. Since V^r is weakly increasing and convex, and using the fact that $X^r \leq \frac{1}{1+\lambda^r}$, it follows that the slope of V^r must approach 0 at all skill levels. **QED**

Lemma A-3 *The righthand derivatives of the two-period profit function V^j, denoted by V^{j+}, satisfy*

$$
\begin{aligned}
V^{j+}\left(\left(\alpha_1^{j\prime}\right)^{-1}\left(X^j\right)\right) &= \left(q^j\right)^2\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(1-X^j\left(1-q^j\right)\right) \\
V^{j+}\left(X^j\right) &= 2q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(1-X^jq^j\left(1-q^j\right)\right).
\end{aligned}
$$

Proof of Lemma A-3: Note that

$$\begin{aligned}\frac{dp_1^j(\alpha)}{d\alpha} &= \frac{dp_0^j(\alpha)}{d\alpha} = q^j\left(1-q^j\right)\\ \frac{d}{d\alpha}\alpha_1^{j\prime}(\alpha)\,p_1^j(\alpha) &= 1-\frac{d}{d\alpha}\alpha_0^{j\prime}(\alpha)\,p_2^j(\alpha) = q^j.\end{aligned}$$

So

$$\begin{aligned}V^{j+}\left(\left(\alpha_1^{j\prime}\right)^{-1}\left(X^j\right)\right) &= q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\\ &\times\left(\frac{dp_1^j(\alpha_0)\,\alpha_1^{j\prime}(\alpha)}{d\alpha}-X^j\frac{dp_1^j(\alpha)}{d\alpha}\right)\\ &= q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(q^j-X^jq^j\left(1-q^j\right)\right)\\ &= \left(q^j\right)^2\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(1-X^j\left(1-q^j\right)\right).\end{aligned}$$

Likewise,

$$\begin{aligned}V^{j+}(X) &= q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\\ &\times\left(1+\frac{d\left(p_1^j(\alpha)\,\alpha_1^{j\prime}(\alpha)+p_0^j(\alpha)\,\alpha_2^{j\prime}(\alpha)\right)}{d\alpha}-X^j\frac{d\left(p_1^j(\alpha)+p_0^j(\alpha)\right)}{d\alpha}\right)\\ &= q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(1+1-2X^jq^j\left(1-q^j\right)\right)\\ &= 2q^j\left(1-q^j\right)\left(h_L^j\left(q^j+\left(1-q^j\right)X^j\right)+h_S^j\left(q^j-q^jX^j\right)\right)\left(1-X^jq^j\left(1-q^j\right)\right).\end{aligned}$$

QED

Proof of Proposition A-1: First, we show $(\alpha_1^{r\prime})^{-1}(X^r) < (\alpha_1^{c\prime})^{-1}(X^c)$. Suppose to the contrary that $(\alpha_1^{r\prime})^{-1}(X^r) \geq (\alpha_1^{c\prime})^{-1}(X^c)$ even as q^r grows small. From Lemma A-3, the righthand derivative $V^{c+}\left((\alpha_1^{c\prime})^{-1}(X^c)\right)$ is bounded away from 0. In contrast, from Lemma A-2 we know $V^{r+}(X^r) \to 0$. It follows that no-one trades the r-asset for q^r sufficiently small. But then $X^r = 0$. If $(\alpha_1^{c\prime})^{-1}(X^c) > 0$ this gives a contradiction, since traders with skill in the interval $\left[0, (\alpha_1^{c\prime})^{-1}(X^c)\right]$ would trade the r-asset in the first period. If instead

$(\alpha_1^{c\prime})^{-1}(X^c) = 0$, then on the one hand, $X^c = 0$ by (A-1); but on the other hand, since $X^c = X^r = 0$, we must have trade in at least one of the assets, implying $X^r + X^c > 0$, and contradicting $X^r + X^c = 0$. This completes the proof of $(\alpha_1^{r\prime})^{-1}(X^r) < (\alpha_1^{c\prime})^{-1}(X^c)$.

Given $(\alpha_1^{r\prime})^{-1}(X^r) < (\alpha_1^{c\prime})^{-1}(X^c)$, we show $(\alpha_1^{r\prime})^{-1}(X^r) > 0$. Suppose to the contrary that $(\alpha_1^{r\prime})^{-1}(X^r) = 0$. So by (A-1), $X^r = 0$. But then traders with skill in $\left[0, (\alpha_1^{c\prime})^{-1}(X^c)\right]$ trade the r-asset immediately, contradicting $X^r = 0$. **QED**

Lemma A-4 *For all q^r small enough, some people trade the r-asset when young.*

Proof of Lemma A-4: Suppose otherwise. Then the only people trading the r-asset are traders who successfully predicted either ω_1^r or ω_0^r when young. It is immediate from (A-3) that $A^{Lr} \to 0$ as $q^r \to 0$. In addition, $(\alpha_0^{r\prime})^{-1}(X^r) \to X^r$, so the interval of skill types $\left[(\alpha_0^{r\prime})^{-1}(X^r), X^r\right]$ who trade when old after successful prediction of ω_0^r when young, grows arbitrarily small. Hence aggregate skill A^r trading the r-asset approaches 0, so that $X^r \to 0$, contradicting Lemma A-1, and completing the proof. **QED**

Proof of Proposition A-2: First, note from (11) that $X^r \leq \frac{1}{\lambda^j+1}$ since $A^r \leq N^r$, and hence X^r stays bounded away from 1.[14]

Second, note that X^c stays bounded away from 0, as follows. Suppose to the contrary that $X^c \to 0$. From Lemma A-3, the slope of V^c is bounded away from 0 for all skill values above $(\alpha_1^{c\prime})^{-1}(X^c)$. In contrast, from Lemma A-2 the slope of V^r approaches 0 as $q^r \to 0$. Hence skill in the c-asset, A^c, is bounded away from 0 as $q^r \to 0$, contradicting $X^c \to 0$.

Suppose that, contrary to the claimed result, $X^c \leq X^r$ even as $q^r \to 0$. So by above, X^c is bounded away from both 0 and 1. Hence

$$
\begin{aligned}
X^c - (\alpha_1^{c\prime})^{-1}(X^c) &= X^c - \frac{X^c q^c}{1 - X^c(1-q^c)} = \frac{(1 - X^c(1-q^c))X^c - q^c X^c}{1 - X^c(1-q^c)} \\
&= \frac{(1-q^c)X^c - (1-q^c)(X^c)^2}{1 - X^c(1-q^c)} = \frac{(1-q^c)X^c(1-X^c)}{1 - X^c(1-q^c)}. \qquad \text{(A-4)}
\end{aligned}
$$

[14] An alternative argument for why X^r is bounded away from 1 is as follows. Suppose instead that $X^r \to 1$ as $q^r \to 0$. Then the fraction of skilled traders who can trade approaches 0, implying $A^r \to 0$, contradicting $X^r \to 1$.

is bounded away from 0. By Lemma A-2, the slope of V^r approaches 0 at all skill levels. On the other hand, from Lemma A-3, the slope of V^c is bounded away from 0 for all skill values above $(\alpha_1^{c\prime})^{-1}(X^c)$. Since $X^c - (\alpha_1^{c\prime})^{-1}(X^c)$ is bounded away from 0, it follows that $V^c(X^c) > \max_\alpha V^r(\alpha)$. So no-one with initial skill above X^c specializes in the r-asset. By the supposition $X^c \leq X^r$, a fortiori no-one with initial skill above X^c specializes in the r-asset. Hence no-one trades the r-asset when young, contradicting Lemma A-4 and completing the proof. **QED**

Proof of Corollary A-1: Immediate from the fact that Lemma A-1 implies that A^r is bounded away from 0. **QED**

Proof of Proposition A-3: It is immediate from (A-3) that $A^{Lr} \to 0$ as $q^r \to 0$. From Lemma A-1, X^r remains bounded away from 0. Hence A^r remains bounded away from 0, completing the proof. **QED**

Proof of Proposition A-4: First, note from (11) that $X^c \leq \frac{1}{\lambda^c+1}$ since $A^c \leq N^c$, and hence X^c stays bounded away from 1.[15]

By Proposition A-2 and Lemma A-1, X^c must also remain bounded away from 0.

Given that X^c is bounded away from both 0 and 1, the same argument as in the proof of Proposition A-2 implies that $V^c(X^c) > \max_\alpha V^r(\alpha)$. So $V^c(\alpha) > V^r(\alpha)$ for all $\alpha \geq X^c$. Since some skilled traders trade the r-asset (Lemma A-4), the curves V^c and V^r must intersect at a skill level strictly below X^c. Moreover, from Lemma A-3 and Lemma A-2, for q^r small the slope of V^c is steeper than the slope of V^r for all skill levels above $(\alpha_1^{c\prime})^{-1}(X^c)$. Hence V^c and V^r intersect exactly once above $\min_{j=r,c}\left\{\left(\alpha_1^{j\prime}\right)^{-1}(X^j)\right\}$, and the intersection point is below X^c. Finally, by Lemma A-4, the intersection point is above X^r. This completes the proof. **QED**

[15] An alternative argument for why X^c is bounded away from 1 is as follows. Suppose instead that $X^c \to 1$ as $q^r \to 0$. Then (by (A-1)) $(\alpha_1^{c\prime})^{-1}(X^c) \to 1$, but then no-one trades the c-asset, contradicting $X^c \to 1$.

References

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