Appendix: Message Games

Let $M = \{m_1, \ldots, m_n\}$ be a finite set of possible messages. Let $\Lambda(M)$ be the set of probability distributions over M, with typical member $(\lambda^1, \ldots, \lambda^n)$. For any $\lambda \in \Lambda(M)$, let suppλ denote the support of λ, i.e., supp$\lambda \equiv \{m_i : \lambda^i > 0\}$.

Write F^i for the distribution of incomes for agents reporting message m_i, and let $G = (F^1, \ldots, F^n, \lambda^1, \ldots, \lambda^n) \in \mathcal{F}^n \times \Lambda(M)$ represent the joint distribution of incomes and messages.

For use below, given a fraction of agents who invest, η, define $G_\eta(\zeta_0, \zeta_1)$ to be the distribution corresponding to the η investors playing mixed-strategy ζ_1 in the message stage and the $1 - \eta$ non-investors playing mixed-strategy ζ_0 in the message stage.

Formally, a tax/transfer policy that uses messages is a function $T(y, m, G)$: an agent's after-transfer income depends on his pre-tax income y, his message m, and the joint distribution over messages and incomes G. As in our main analysis, we assume that T is continuous in its arguments.

We consider the following investment game. The government announces a tax policy. Agents make investment decisions and submit messages to the government. The government observes incomes and implements its tax policy. As before, agents do not observe each others' actions.

Taking η as given, and with some abuse of language, we will say that message strategies $(\lambda_0, \lambda_1) \in \Lambda(M) \times \Lambda(M)$ constitute an *equilibrium given* η if for $a \in \{0, 1\}$ and $m \in$suppλ_a, then $m \in \arg\max_{m'} T(f_a(\eta), m', G(\lambda_0, \lambda_1))$. That is, given the message-income distribution $G(\lambda_0, \lambda_1)$, every message reported with strictly positive probability by investors (respectively, non-investors) is a best response given the investor's (respectively, non-investor's) income level. Standard arguments imply that for any η an equilibrium given η exists.[21]

An equilibrium is an investment level η and a pair of reporting strategies (λ_0, λ_1) such that (λ_0, λ_1) is an equilibrium given η, and such that

$$\begin{aligned} \max_{m' \in M} T\left(f_1(\eta), m', G(\lambda_0, \lambda_1)\right) &\geq \max_{m' \in M} T\left(f_0(\eta), m', G(\lambda_0, \lambda_1)\right) \text{ if } \eta \in (0,1] \\ \max_{m' \in M} T\left(f_1(\eta), m', G(\lambda_0, \lambda_1)\right) &\leq \max_{m' \in M} T\left(f_0(\eta), m', G(\lambda_0, \lambda_1)\right) \text{ if } \eta \in [0,1). \end{aligned}$$

Proposition 4 *Suppose that $\eta_1, \ldots, \eta_k$ are the equilibrium investment levels given T. Then there is a transfer scheme $\hat{T}$ that makes no use of messages such that $\eta_1, \ldots, \eta_k$ are the equilibrium investment levels given $\hat{T}$ also.*

Proof: Define $\hat{T}$ as follows. For each investment level $\eta \in \{\eta_1, \ldots, \eta_k\}$ let (λ_0, λ_1) be a pair of message strategies that together with investment level η constitute an equilibrium, and define $\hat{T}(f_a(\eta), F_\eta) = \max_{m \in M} T(f_a(\eta), m, G_\eta(\lambda_0, \lambda_1))$. For other investment levels $\eta \notin \{\eta_1, \ldots, \eta_k\}$, choose a pair of message strategies (λ_0, λ_1) that are an equilibrium given η. That is, if agents were somehow "stuck" at investment level η, and knew this, (λ_0, λ_1) would be equilibrium message strategies. Again, define $\hat{T}(f_a(\eta), F_\eta) = \max_{m \in M} T(f_a(\eta), m, G_\eta(\lambda_0, \lambda_1))$.[22]

By construction, for $i = 1, \ldots, k$, investment level η_i is indeed an equilibrium given the new policy $\hat{T}$.

[21] To see this, define a correspondence $Z : \Lambda(M) \times \Lambda(M) \to\to \Lambda(M) \times \Lambda(M)$ by

$$\begin{aligned} Z(\lambda_0, \lambda_1) = &\{(\zeta_0, \zeta_1) \in \Lambda(M) \times \Lambda(M) \text{ such that for } a = 0, 1, \\ &\text{if } m \in \text{supp}\zeta_a \text{ then } m \in \arg\max_{m'} T\left(f_a(\eta), m', G(\lambda_0, \lambda_1)\right).\} \end{aligned}$$

That is, for strategies λ_0 and λ_1, the correspondance Z gives the strategies that have support over messages that are best responses given $G(\lambda_0, \lambda_1)$. Trivially $\Lambda(M) \times \Lambda(M)$ is compact, convex and non-empty. $Z(\lambda_0, \lambda_1)$ is non-empty and convex for all $(\lambda_0, \lambda_1) \in \Lambda(M) \times \Lambda(M)$. Finally, given the continuity of T, the correspondance Z is closed. Kakutani's fixed point theorem thus applies, and implies the existence of some (λ_0, λ_1) such that $(\lambda_0, \lambda_1) \in Z(\lambda_0, \lambda_1)$. The strategies (λ_0, λ_1) constitute an equilibrium.

[22] The policy $\hat{T}$ can be defined for all η provided that the income distribution differs for all η. This condition is generically statified.

Conversely, consider any other investment level $\eta \notin \{\eta_1, \ldots, \eta_k\}$. Suppose for now that $\eta \in (0,1)$. Since η is not an equilibrium investment level, it must be the case that in any candidate equilibrium of the message-investment game either investment or non-investment gives a strictly higher payoff. Formally, for *any* message strategies (λ_0, λ_1) that are an equilibrium given η,

$$\max_{m' \in M} T\left(f_1(\eta), m', G(\lambda_0, \lambda_1)\right) \neq \max_{m' \in M} T\left(f_0(\eta), m', G(\lambda_0, \lambda_1)\right).$$

But from this it follows that

$$\hat{T}(f_1(\eta), F_\eta) \neq \hat{T}(f_0(\eta), F_\eta),$$

so that η is not an equilibrium given $\hat{T}$ either. The cases $\eta = 0$ and $\eta = 1$ follow similarly. **QED**