Supplementary Online Appendix

B.1 Notes on the definition of Lehmann informativeness

I have defined Lehmann informativeness in terms of the function $I(x,\theta) : \mathcal{X}(\theta;\kappa_2) \to \mathcal{X}(\theta;\kappa_1)$, defined by

$$F(I(x,\theta)|\theta;\kappa_1) = F(x|\theta;\kappa_2).$$

The condition is:

[L-I] For any $x \in \mathcal{X}(\kappa_2)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1;\kappa_2) \cap \mathcal{X}(\theta_2;\kappa_2)$, $I(x,\theta_1) \geq I(x,\theta_2)$.

Typically, the definition is instead stated in terms of the function $J(x,\theta) : \mathcal{X}(\theta;\kappa_1) \to \mathcal{X}(\theta;\kappa_2)$, defined by

$$F(x|\theta;\kappa_1) = F(J(x,\theta)|\theta;\kappa_2).$$

The condition is then:

[L-J] For any $x \in \mathcal{X}(\kappa_1)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1;\kappa_1) \cap \mathcal{X}(\theta_2;\kappa_1)$, $J(x,\theta_2) \geq J(x,\theta_1)$.

Note that I and J are inverses. Specifically, for any $x \in \mathcal{X}(\theta;\kappa_1)$, $I(J(x,\theta)) = x$, and for any $x \in \mathcal{X}(\theta;\kappa_2)$, $J(I(x,\theta)) = x$. These statements make use of the fact that both I and J are strictly increasing in their first argument (by Property 1).

B.1.1 The advantage of stating the Lehmann informativeness in terms of [L-I]

The two formulations are equivalent under mild regularity conditions. The property actually used in the proof of Proposition 1 is that I is decreasing. Given non-equivalence under "pathological" conditions, it is easiest to simply state the definition in terms of [L-I].

B.1.2 Equivalence under many conditions

When the supports $\mathcal{X}(\theta;\kappa)$ are well-behaved, in terms of not varying too much in θ, the two definitions are equivalent.

Specifically:

Lemma 2 *If $\mathcal{X}(\theta_1;\kappa_1) \cap \mathcal{X}(\theta_2;\kappa_1) \neq \emptyset$ for all $\theta_1, \theta_2 \in \Theta$ then [L-J] implies [L-I].*

Lemma 3 *If $\mathcal{X}(\theta_1;\kappa_2) \cap \mathcal{X}(\theta_2;\kappa_2) \neq \emptyset$ for all $\theta_1, \theta_2 \in \Theta$ then [L-I] implies [L-J].*

Note, moreover, that the global non-empty intersection properties can be considerably weakened to ones that hold only locally. For transparency, I state the proof for the global property.

Proof of Lemma 2: Suppose [L-J] holds, but [L-I] is violated, i.e., for some $x \in \mathcal{X}(\kappa_2)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2)$, $I(x, \theta_2) > I(x, \theta_1)$.

Certainly $I(x, \theta_1) \in \mathcal{X}(\theta_1; \kappa_1)$ and $I(x, \theta_2) \in \mathcal{X}(\theta_2; \kappa_1)$. Since $\mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1) \neq \emptyset$, it follows from Property 1 that there exists $x_0 \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$ such that

$$I(x, \theta_2) \geq x_0 \geq I(x, \theta_1),$$

with at least one of the two inequalities strict. But then

$$x = J(I(x, \theta_2), \theta_2) \geq J(x_0, \theta_2) \geq J(x_0, \theta_1) \geq J(I(x, \theta_1), \theta_1) = x,$$

with at least one of the first and third inequalities being strict. The contradiction completes the proof.

Proof of Lemma 3: Suppose [L-I] holds, but [L-J] is violated, i.e., for some $x \in \mathcal{X}(\kappa_1)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$, $J(x, \theta_2) < J(x, \theta_1)$.

Certainly $J(x, \theta_1) \in \mathcal{X}(\theta_1; \kappa_2)$ and $J(x, \theta_2) \in \mathcal{X}(\theta_2; \kappa_2)$. Since $\mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2) \neq \emptyset$, it follows from Property 1 that there exists $x_0 \in \mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2)$ such that

$$J(x, \theta_2) \leq x_0 \leq J(x, \theta_1),$$

with at least one of the two inequalities strict. But then

$$x = I(J(x, \theta_2), \theta_2) \leq I(x_0, \theta_2) \leq I(x_0, \theta_1) \leq I(J(x, \theta_1), \theta_1) = x,$$

with at least one of the first and third inequalities being strict. The contradiction completes the proof.

B.1.3 A simple example in which [L-I] holds but [L-J] is violated

Consider a case in which $\Theta = \{\theta_1, \theta_2\}$, with $\theta_2 > \theta_1$, $\mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1) \neq \emptyset$ but $\mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2) = \emptyset$, and $\mathcal{X}(\theta_2; \kappa_2) < \mathcal{X}(\theta_1; \kappa_2)$. (Since these sets don't intersect, this ordering is unambiguous.)

In this case, [L-I] holds vacuously, while trivially, if $x \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$, then $J(x, \theta_2) < J(x, \theta_1)$, so that [L-J] is violated.

Note, moreover, that this simple case is one in which the regime κ_2 is unambiguously more informative than regime κ_1, since in regime κ_2 the observation of X fully reveals the value of $\theta \in \{\theta_1, \theta_2\}$, while this isn't the case in regime κ_1.

B.2 Detailed calculations used in subsection 2.1

(I) Consider $\bar{u}(x;t) = u\left(W(t) + \bar{R}x\right)$ and $\underline{u}(x;t) = u\left(W(t) + \underline{R}x\right)$, where $W'(t) < 0$, $\underline{R} < 0 < \bar{R}$, and u features decreasing absolute risk aversion (DARA).

In this case,

$$\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right) = \frac{\partial}{\partial t}\ln\left(-\frac{\bar{R}}{\underline{R}}\frac{u'\left(W(t)+\bar{R}x\right)}{u'\left(W(t)+\bar{R}x\right)}\right) = W'(t)\frac{u''\left(W(t)+\bar{R}x\right)}{u'\left(W(t)+\bar{R}x\right)} - W'(t)\frac{u''\left(W(t)+\underline{R}x\right)}{u'\left(W(t)+\underline{R}x\right)}.$$

Hence DARA implies (6). Moreover, DARA further implies that $-\frac{u''\left(W(t)+\bar{R}x\right)}{u'\left(W(t)+\bar{R}x\right)}$ is decreasing in x and $-\frac{u''\left(W(t)+\underline{R}x\right)}{u'\left(W(t)+\underline{R}x\right)}$ is increasing in x, so that (7) holds.

(II) Consider $\bar{u}(x;t) = (1-t)\,u\left(\bar{W}+\bar{R}x\right) + tu\left(\underline{W}+\bar{R}x\right)$ and $\underline{u}(x;t) = (1-t)\,u\left(\bar{W}+\underline{R}x\right) + tu\left(\underline{W}+\underline{R}x\right)$, where $\underline{W} < \bar{W}$, $\underline{R} < 0 < \bar{R}$, and u features DARA. In this case,

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right) &= \frac{u'\left(\underline{W}+\bar{R}x\right) - u'\left(\bar{W}+\bar{R}x\right)}{(1-t)\,u'\left(\bar{W}+\bar{R}x\right) + tu'\left(\underline{W}+\bar{R}x\right)} - \frac{u'\left(\underline{W}+\underline{R}x\right) - u'\left(\bar{W}+\underline{R}x\right)}{(1-t)\,u'\left(\bar{W}+\underline{R}x\right) + tu'\left(\underline{W}+\underline{R}x\right)} \\ &= \frac{\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)} - 1}{(1-t) + t\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)}} - \frac{\frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)} - 1}{(1-t) + t\frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)}}. \end{aligned} \tag{29}$$

So (6) holds, since the expression $\frac{y-1}{1-t+ty}$ is increasing in y, and $\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)} < \frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)}$ by DARA, since DARA implies that $\frac{u'\left(\underline{W}+y\right)}{u'\left(\bar{W}+y\right)}$ is decreasing in y. These same observations also imply that the first term in (29) is decreasing in x while the second term is increasing in x, so that (7) holds.

B.3 Detailed calculations used in subsection 2.2

B.3.1 Demand decreasing in price, $q_x < 0$

Writing the FOC (11) explicitly gives

$$(1-\psi)\left(\bar{R}-x\right)u'\left(q\left(\bar{R}-x\right)\right) + \psi\left(\underline{R}-x\right)u'\left(q\left(\underline{R}-x\right)\right) = 0. \tag{30}$$

Since $U_{qq} < 0$, this has at most one solution in q. The derivative of the RHS of (30) with respect to x is

$$-\left((1-\psi)\,u'\left(q\left(\bar{R}-x\right)\right) + \psi u'\left(q\left(\underline{R}-x\right)\right)\right)$$

$$- \quad q\left((1-\psi)\left(\bar{R}-x\right)u''\left(q\left(\bar{R}-x\right)\right)+\psi\left(\underline{R}-x\right)u''\left(q\left(\underline{R}-x\right)\right)\right). \tag{31}$$

The first term is strictly negative. The second term equals

$$q\left(\left|\frac{u''\left(q\left(\bar{R}-x\right)\right)}{u'\left(q\left(\bar{R}-x\right)\right)}\right|(1-\psi)\left(\bar{R}-x\right)u'\left(q\left(\bar{R}-x\right)\right)+\left|\frac{u''\left(q\left(\underline{R}-x\right)\right)}{u'\left(q\left(\underline{R}-x\right)\right)}\right|\psi\left(\underline{R}-x\right)u'\left(q\left(\underline{R}-x\right)\right)\right).$$

By DARA,

$$q\left|\frac{u''\left(q\left(\bar{R}-x\right)\right)}{u'\left(q\left(\bar{R}-x\right)\right)}\right|\leq q\left|\frac{u''\left(q\left(\underline{R}-x\right)\right)}{u'\left(q\left(\underline{R}-x\right)\right)}\right|,$$

and so expression (31) is strictly below

$$q\left|\frac{u''\left(q\left(\underline{R}-x\right)\right)}{u'\left(q\left(\underline{R}-x\right)\right)}\right|\left((1-\psi)\left(\bar{R}-x\right)u'\left(q\left(\bar{R}-x\right)\right)+\psi\left(\underline{R}-x\right)u'\left(q\left(\underline{R}-x\right)\right)\right)=q\left|\frac{u''\left(q\left(\underline{R}-x\right)\right)}{u'\left(q\left(\underline{R}-x\right)\right)}\right|U_q,$$

which is simply 0 at the informed agent's optimal demand $q(x,\theta,\kappa)$. Hence an increase in x must strictly reduce $q(x,\theta,\kappa)$.

B.3.2 Derivation of (14)

By straightforward substitution,

$$\begin{aligned}
\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\bigg|_{x=x(\theta,t,\kappa)} &= \frac{\partial}{\partial\theta}\ln\left(\frac{U_{qx}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right)}{U_{q\kappa}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right)}\right)\bigg|_{x=x(\theta,t,\kappa)} \\
&= \frac{U_{qx\theta}+q_\theta U_{qqx}}{U_{qx}}-\frac{U_{q\theta\kappa}+q_\theta U_{qq\kappa}}{U_{q\kappa}} \\
&= \frac{U_{qx\theta}-\frac{U_{q\theta}}{U_{qq}}U_{qqx}}{U_{qx}}-\frac{U_{q\theta\kappa}-\frac{U_{q\theta}}{U_{qq}}U_{qq\kappa}}{U_{q\kappa}}.
\end{aligned}$$

Recall that θ and κ enter U only via the function ψ, and moreover, U is linear in ψ. Accordingly, write U_ψ etc to denote the derivative of U with respect to ψ. Hence $U_{q\theta}=\psi_\theta U_{q\psi}$, $U_{qx\theta}=\psi_\theta U_{qx\psi}$, $U_{q\kappa}=\psi_\kappa U_{q\psi}$, $U_{qq\kappa}=\psi_\kappa U_{qq\psi}$, and $U_{q\theta\kappa}=\psi_\theta\psi_\kappa U_{q\psi\psi}+\psi_{\theta\kappa}U_{q\psi}=\psi_{\theta\kappa}U_{q\psi}$. Hence

$$\begin{aligned}
\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\bigg|_{x=x(\theta,t,\kappa)} &= \frac{\psi_\theta U_{qx\psi}-\psi_\theta\frac{U_{q\psi}}{U_{qq}}U_{qqx}}{U_{qx}}-\frac{\psi_{\theta\kappa}U_{q\psi}-\psi_\theta\psi_\kappa\frac{U_{q\psi}}{U_{qq}}U_{qq\psi}}{\psi_\kappa U_{q\psi}} \\
&= -\psi_\theta\left(\frac{U_{q\psi}U_{qqx}-U_{qq}U_{qx\psi}}{U_{qq}U_{qx}}+\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa}-\frac{U_{qq\psi}}{U_{qq}}\right).
\end{aligned}$$

To ease notation, define $\bar{u}' \equiv u'\left(q\left(\bar{R}-x\right)\right)$ and $\underline{u}' = u'\left(q\left(\underline{R}-x\right)\right)$, with analogous definitions for higher order derivatives. Straightforward differentiation yields

$$\begin{aligned}
U_q &= (1-\psi)\left(\bar{R}-x\right)\bar{u}' + \psi\left(\underline{R}-x\right)\underline{u}', \\
U_{qx} &= -(1-\psi)\bar{u}' - \psi\underline{u}' - (1-\psi)\left(\bar{R}-x\right)q\bar{u}'' - \psi\left(\underline{R}-x\right)q\underline{u}'', \\
U_{qq} &= (1-\psi)\left(\bar{R}-x\right)^2\bar{u}'' + \psi\left(\underline{R}-x\right)^2\underline{u}'', \\
U_{q\psi} &= -\left(\left(\bar{R}-x\right)\bar{u}' - \left(\underline{R}-x\right)\underline{u}'\right), \\
U_{qx\psi} &= \bar{u}' - \underline{u}' + \left(\bar{R}-x\right)q\bar{u}'' - \left(\underline{R}-x\right)q\underline{u}'', \\
U_{qq\psi} &= -\left(\left(\bar{R}-x\right)^2\bar{u}'' - \left(\underline{R}-x\right)^2\underline{u}''\right), \\
U_{qqx} &= -2(1-\psi)\left(\bar{R}-x\right)\bar{u}'' - 2\psi\left(\underline{R}-x\right)\underline{u}'' - (1-\psi)\left(\bar{R}-x\right)^2 q\bar{u}''' - \psi\left(\underline{R}-x\right)^2 q\underline{u}'''.
\end{aligned}$$

I first establish that

$$\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} - \frac{U_{qq\psi}}{U_{qq}} > \frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} - \frac{1}{\psi}. \tag{32}$$

Evaluating,

$$\begin{aligned}
-\frac{U_{qq\psi}}{U_{qq}} &= \frac{\left(\bar{R}-x\right)^2\bar{u}'' - \left(\underline{R}-x\right)^2\underline{u}''}{\left(\bar{R}-x\right)^2\bar{u}'' - \psi\left(\left(\bar{R}-x\right)^2\bar{u}'' - \left(\underline{R}-x\right)^2\underline{u}''\right)} \\
&= \frac{\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - 1}{\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - \psi\left(\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - 1\right)}.
\end{aligned}$$

Note the function $\frac{y-1}{y-\psi(y-1)}$ is increasing in y, since $y - \psi(y-1) - (y-1)(1-\psi) = 1 > 0$. Hence the function $\frac{y-1}{y-\psi(y-1)}$ varies from $-\frac{1}{\psi}$ to $\frac{1}{1-\psi}$ as y varies from 0 to ∞. Consequently,

$$-\frac{U_{qq\psi}}{U_{qq}} > -\frac{1}{\psi},$$

establishing (32).

Second, I consider the term

$$\frac{U_{q\psi}U_{qqx} - U_{qq}U_{qx\psi}}{U_{qq}U_{qx}}. \tag{33}$$

As $s(0), s(1) \to 0$, the equilibrium value of q approaches 0 for all realizations of t, so that $\frac{\bar{u}'}{\underline{u}'}, \frac{\bar{u}''}{\underline{u}''}, \frac{\bar{u}'''}{\underline{u}'''} \to 1$ and $x \to (1-\psi)\bar{R} + \psi\underline{R}$. Hence $U_{qqx} \to 0$ and $U_{qx\psi} \to 0$ while the other terms in (33) remain bounded away from 0, implying that (33) converges to 0, and establishing

(14).

B.3.3 Comparison of log-submodularity of ψ and log-supermodularity of the likelihood ratio $\frac{1-\psi}{\psi}$

Note that log-submodularity of ψ implies log-supermodularity of the likelihood ratio $\frac{1-\psi}{\psi}$, as follows. Log supermodularity of $\frac{1-\psi}{\psi}$ is equivalent to

$$\frac{\partial}{\partial\theta}\left(\frac{-\psi_\kappa}{1-\psi}-\frac{\psi_\kappa}{\psi}\right)\geq 0,$$

i.e.,

$$\frac{\psi_{\kappa\theta}(1-\psi)+\psi_\kappa\psi_\theta}{(1-\psi)^2}+\frac{\psi_{\kappa\theta}\psi-\psi_\kappa\psi_\theta}{\psi^2}\leq 0,$$

i.e.,

$$(1-\psi)\,\psi\psi_{\kappa\theta}+\left(\psi^2-(1-\psi)^2\right)\psi_\kappa\psi_\theta\leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta}+\frac{2\psi-1}{1-\psi}\psi_\kappa\psi_\theta\leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta}-\psi_\kappa\psi_\theta+\left(\frac{2\psi-1}{1-\psi}+1\right)\psi_\kappa\psi_\theta\leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta}-\psi_\kappa\psi_\theta+\frac{\psi}{1-\psi}\psi_\kappa\psi_\theta\leq 0.$$

B.4 Detailed calculations used in subsection 2.3

B.4.1 Verification of equilibrium price (17)

Let $\xi(\theta,t)$ be the value of $x>0$ that solves

$$\theta-x-\lambda(t)+\frac{A}{x}=0. \tag{34}$$

Solving explicitly,

$$x^2+x\left(\lambda(t)-\theta\right)-A=0.$$

Focusing on the positive-valued solution, it follows that

$$\xi(\theta,t)=\frac{1}{2}\left(\theta-\lambda(t)+\sqrt{(\theta-\lambda(t))^2+4A}\right).$$

Note that ξ is strictly increasing in θ and strictly decreasing in t. Moreover,

$$\xi\left(\frac{A}{\lambda(t)}, t\right) = \frac{A}{\lambda(t)}.$$

If $\theta > \frac{A}{\lambda(t)}$ then the conjectured price is $\xi(\theta, t) > \frac{A}{\lambda(t)}$. Liquidity demand is $-\lambda(t) + \frac{A}{\xi(\theta,t)} < 0$. Since $\xi(\theta, t)$ solves (34), it follows that $\theta > \xi(\theta, t)$. Hence informed demand is $\theta - \xi(\theta, t)$. Since $\xi(\theta, t)$ solves (34) it follows that the market-clearing condition (16) holds.

If $\theta \in \left[\frac{A}{\lambda(t)} - K, \frac{A}{\lambda(t)}\right]$ then the conjectured price is $\xi(\theta, t) = \frac{A}{\lambda(t)}$. Liquidity demand is hence $-\lambda(t) + \frac{A}{\xi(\theta,t)} = 0$. Since $\theta \in [\xi(\theta, t) - K, \xi(\theta, t)]$, informed demand is also 0. Hence the market-clearing condition (16) holds.

If $\theta < \frac{A}{\lambda(t)} - K$ then the conjectured price is $\xi(\theta + K, t) < \frac{A}{\lambda(t)}$. Liquidity demand is $-\lambda(t) + \frac{A}{\xi(\theta,t)} > 0$. Since $\xi(\theta + K, t)$ solves

$$(\theta + K) - x - \lambda(t) + \frac{A}{x} = 0 \tag{35}$$

it follows that $\theta + K < \xi(\theta + K, t)$. Hence informed demand is $\theta - \xi(\theta, t) + K$. Since $\xi(\theta, t)$ solves (35) it follows that the market-clearing condition (16) holds.

B.4.2 The equilibrium price satisfies SCP

I next show that the equilibrium price $x(\theta, t, K)$, which coincides with the quantile function, satisfies SCP. Let $\theta_1, \theta_2 \geq \theta_1$, $t_1, t_2 \geq t_1$ and $K_1 = K(\kappa_1)$ be such that $x(\theta_2, t_2, K_1) \geq x(\theta_1, t_1, K_1)$, and consider $\kappa_2 > \kappa_1$. I establish that at $K_2 = K(\kappa_2) < K_1$, $x(\theta_2, t_2, K_2) \geq x(\theta_1, t_1, K_2)$, with strict inequality if $x(\theta_2, t_2, K_1) > x(\theta_1, t_1, K_1)$.

First, note that the result is immediate if $t_2 = t_1$, since x is weakly increasing in θ, and moreover, the interval over which x is constant in θ strictly shrinks as K falls from K_1 to K_2. So for the remainder of the proof assume $t_2 > t_1$.

Second, the result is also immediate if $\theta_2 \geq \frac{A}{\lambda(t_2)} - K_2$, since in this case $x(\theta_2, t_2, K_2) = x(\theta_2, t_2, K_1)$ while $x(\theta_1, t_1, K_2) \leq x(\theta_1, t_1, K_1)$. So for the remainder of the proof assume $\theta_2 < \frac{A}{\lambda(t_2)} - K_2$. Hence $\theta_1 < \frac{A}{\lambda(t_1)} - K_2$ also.

Third: Given $t_2 > t_1$ and $\theta_2 < \frac{A}{\lambda(t_2)}$ it follows that $\theta_1 < \frac{A}{\lambda(t_1)} - K_1$. To see this, suppose to the contrary that $\theta_1 \geq \frac{A}{\lambda(t_1)} - K_1$. Then $x(\theta_1, t_1, K_1) = \frac{A}{\lambda(t_1)} > \frac{A}{\lambda(t_2)} \geq x(\theta_2, t_2, K_1)$, a contradiction.

Fourth: Given $\theta_1 < \frac{A}{\lambda(t_1)} - K_1$, it follows that $\theta_2 - \lambda(t_2) \geq \theta_1 - \lambda(t_1)$, as follows. By definition, $x(\theta_1, t_1, K_1)$ solves

$$(\theta_1 + K_1) - x - \lambda(t_1) + \frac{A}{x} = 0.$$

If $\theta_2 \leq \frac{A}{\lambda(t_2)} - K_1$ then the result is immediate from $x(\theta_1, t_1, K_1) \leq x(\theta_2, t_2, K_1)$. If instead $\theta_2 > \frac{A}{\lambda(t_2)} - K_1$ then note that, at $x = \frac{A}{\lambda(t_2)}$,

$$(\theta_2 + K_1) - x - \lambda(t_2) + \frac{A}{x} > 0,$$

and the result again follows from $x(\theta_1, t_1, K_1) \leq x(\theta_2, t_2, K_1) = \frac{A}{\lambda(t_2)}$.

Finally: Given $\theta_2 - \lambda(t_2) \geq \theta_1 - \lambda(t_1)$ the result is immediate from the fact that $x(\theta, t, K_2)$ solves (35) for $(\theta, t) = (\theta_1, t_1), (\theta_2, t_2)$ and $K = K_2$.