Ordering information content using the quantile function*

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Abstract

Lehmann's (1988) ordering of information content is equivalent to a single-crossing property of the quantile function. This equivalence considerably aids the application of Lehmann's ordering.

The information content of prices and agents' actions is central to many areas of economics. Blackwell (1953) develops a very general notion of information content: a variable X is more informative than a variable Y if a decisionmaker would prefer to observe X rather than Y, regardless of the decision problem faced. However, Blackwell's ordering fails to rank many cases of interest. For example, Lehmann (1988) shows that, surprisingly, Blackwell's ordering fails to rank the amount of information about a variable θ that is is conveyed by the family of random variables $X_\kappa = \theta + \frac{\nu}{\kappa}$, where ν is uniformly distributed over $[-1, 1]$.

Lehmann (1988) proposes an alternative notion of information content that ranks more cases than Blackwell's, including the example just given. Lehmann's ordering is stated in a way that is likely to be intuitive to economists: rather than insisting that an arbitrary decisionmaker prefer to observe X rather than Y, Lehmann considers only monotone decision problems, i.e., those in which the decision of a fully informed decisionmaker would be monotone in the underlying state variable. However, Lehmann's ordering is stated in a way that, at first sight, it hard operationalize.

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The main result in this short paper is that Lehmann's ordering is equivalent to a single-crossing property of the quantile function. Although simple, this equivalence has not previously been noted.[1] Exploiting this equivalence considerably aids the application of Lehmann's ordering, especially in differentiable cases. In particular, by working with the Spence-Mirrlees formulation of single-crossing, it is possible to compare the information content of prices and actions without fully solving for prices or optimal actions. The equivalence with single-crossing also provides a geometric intuition for why Lehmann's ordering captures information content.

1 Lehmann-informativeness and the single-crossing property of the quantile function

Lehmann's ordering is as follows. A decisionmaker cares about the realization of a state variable $\theta \in \Theta$, but doesn't observe θ directly. Instead, the decisionmaker observes the realization of a real-valued random variable X, whose distribution depends on θ. The distribution of X also depends on a "regime," which is indexed by the parameter κ, and known to the decisionmaker. The interpretation of the regime κ depends on the application; see Section 2 for examples. In the simple case above of $X = \theta + \frac{\nu}{\kappa}$, the regime κ is simply a scaling parameter that controls the variance of the "noise" term $\frac{\nu}{\kappa}$.

Let $F(\cdot|\theta;\kappa)$ be the distribution of X conditional on θ in regime κ, with $\mathcal{X}(\theta;\kappa)$ denoting the corresponding support. Define $\mathcal{X}(\kappa) \equiv \bigcup_{\theta\in\Theta} \mathcal{X}(\theta;\kappa)$. For any $x \in \mathcal{X}(\kappa)$, define $\Theta(x;\kappa)$ as the set of states such that x lies in the support of X, i.e., $\Theta(x;\kappa) = \left\{\tilde{\theta} : x \in \mathcal{X}\left(\tilde{\theta};\kappa\right)\right\}$.

Lehmann's ordering compares how much information X conveys about θ in two alternate regimes κ_1 and κ_2. Define the function $I(\cdot,\theta;\kappa_1,\kappa_2) : \mathcal{X}(\theta;\kappa_2) \to \mathcal{X}(\theta;\kappa_1)$ by

$$F\left(I\left(x,\theta;\kappa_1,\kappa_2\right)|\theta;\kappa_1\right) = F\left(x|\theta;\kappa_2\right). \tag{1}$$

Definition 1 *X is a more Lehmann-informative in regime κ_2 than κ_1 if for all $x \in \mathcal{X}(\kappa_2)$, the function $I(x,\theta;\kappa_1,\kappa_2)$ is weakly decreasing in $\theta \in \Theta(x;\kappa_2)$.*

Remark: Definition 1 differs slightly from Lehmann's original definition, which is that the

[1]For contributions within economics on Lehmann's ordering, see Persico (2000), Bergemann and Valimaki (2002), Jewitt (2007), Quah and Strulovici (2009), Athey and Levin (2018), Chi and Choi (2019), Li and Zhou (2020), and Kim (2023). In particular, Chi and Choi establish equivalence between Lehmann's ordering and the usefulness of performance measures in agency problems; and as an intermediate step, establish equivalence between Lehmann's ordering and single-crossing in the *difference* in distribution functions; while Kim establishes equivalence between Lehmann's ordering and a weaker version of Blackwell's garbling condition.

inverse of I with respect to x is weakly increasing in θ.[2] Under mild regularity conditions on how the support $\mathcal{X}(\theta;\kappa)$ varies with the underlying state θ, the two conditions are equivalent (see online appendix). Definition 1 has the advantage of being the condition that is used in the proof of Proposition 1, and avoids the need to impose further conditions on how the support $\mathcal{X}(\theta;\kappa)$ varies with θ.

1.1 Lehmann-informativeness and decision problems

Lehmann-informativeness is of interest because it ranks outcomes in a particular class of decision problems. Specifically, consider a decisionmaker who must select $b \in B \subset \Re$. The decisionmaker's objective is to choose b to maximize an objective $V(b,\theta)$, which is continuous in b. The decisionmaker does not observe θ directly, and instead observes only X, as described above.

The objective V satisfies the single-crossing property (SCP, Milgrom and Shannon (1994)) in (b,θ). Hence the decision problem is monotone, in the sense that a decisionmaker who were counterfactually fully informed about θ would choose higher values of b when θ is higher.

To allow for cases in which the choice set B in non-compact, I impose the following relatively mild assumption on how V behaves for low and high choices of $b \in B$: There exist $\underline{\theta}$, $\bar{\theta} \geq \underline{\theta}$, $\underline{b}$ and $\bar{b}$ such that if $\theta \leq \underline{\theta}$ then $V(\cdot,\theta)$ is weakly decreasing for $b \geq \bar{b}$, and if $\theta \geq \bar{\theta}$ then $V(\cdot,\theta)$ is weakly increasing for $b \leq \underline{b}$.

Lehmann (1988) and Quah and Strulovici (2009) establish that if X is more Lehmann-informative in regime κ_2 than κ_1, then the decisionmaker is better off in regime κ_2 than in κ_1.[3] Both papers restrict attention to the case in which the support of X is independent of the realization of θ. To facilitate applications, Proposition 1 below represents a modest generalization of these previous results to the case in which the support of X potentially depends on θ, and in which the decisionmaker's action space is non-compact. For convenience, I state the following two properties used in Proposition 1 separately.

First, the distribution function of X is continuous and strictly increasing:

Property 1 *For all states θ and regimes κ, the support $\mathcal{X}(\theta;\kappa)$ is an interval, and the distribution function $F(\cdot|\theta;\kappa)$ is continuous and strictly increasing over $\mathcal{X}(\theta;\kappa)$, with* $\inf_{x\in\mathcal{X}(\theta;\kappa)} F(x|\theta;\kappa) = 0$ *and* $\sup_{x\in\mathcal{X}(\theta;\kappa)} F(x|\theta;\kappa) = 1$.

Second, any shift in the support of X across regimes satisfies the following mild restriction:

[2]The inverse of I is the function $J(\cdot,\theta;\kappa_1,\kappa_2) : \mathcal{X}(\theta;\kappa_1) \rightarrow \mathcal{X}(\theta;\kappa_2)$ defined by $F(x|\theta;\kappa_1) = F(J(x,\theta;\kappa_1,\kappa_2)|\theta;\kappa_2)$.

[3]Lehmann (1988) imposes slightly different assumptions on the decisionmaker's objective.

Property 2 *Let $\theta \in \Theta$ and $\kappa, \tilde{\kappa}$ be alternate regimes. The support $\mathcal{X}(\theta;\kappa)$ is unbounded above (respectively, below) if and only if $\mathcal{X}(\theta;\tilde{\kappa})$ is unbounded above (below).*

Properties 1 and 2 ensure that the function I, defined in (1) and on which the Lehmann ordering is based, is well-defined and a bijection.

Proposition 1 *Let Properties 1 and 2 hold. If X is more Lehmann-informative in regime κ_2 than κ_1, and $\zeta : \mathcal{X}(\kappa_1) \to B$ is a weakly increasing function, then there exists $\phi : \mathcal{X}(\kappa_2) \to B$ such that, for all θ, $V(\phi(X), \theta)$ in regime κ_2 first-order stochastically dominates $V(\zeta(X), \theta)$ in regime κ_1.*

Because Proposition 1 is close to existing results, I relegate its proof to Appendix B.

As Quah and Strulovici (2009) emphasize, Lehmann-informativeness implies an improvement in the decisionmaker's payoff in a very robust sense, in that Proposition 1 is completely independent of the decisionmaker's prior beliefs of θ. Moreover, Li and Zhou (2020) relate Lehmann-informativeness to a utility improvement for uncertainty-averse decisionmakers.

Proposition 1 is predicated on the decisionmaker's action being weakly increasing in X in the initial regime κ_1. Lehmann (1988) and Quah and Strulovici (2009) each give sufficient conditions for this. In both cases, the conditions include that the monotone likelihood ratio property (MLRP) holds.[4]

1.2 Equivalence of Lehmann-informativeness to the quantile function satisfying the SCP

The main result in this paper is that Lehmann-informativeness is equivalent to a single-crossing property of the quantile function $F^{-1}(1-t|\theta;\kappa)$, which specifies the top t percentile of X.

Given Property 1, for all $t \in (0,1)$ the quantile function $F^{-1}(1-t|\theta;\kappa)$ is uniquely defined. In addition, define $F^{-1}(0|\theta;\psi) = \inf \mathcal{X}(\theta;\psi)$ and $F^{-1}(1|\theta;\psi) = \sup \mathcal{X}(\theta;\psi)$, with the understanding that if $\mathcal{X}(\theta;\psi)$ is unbounded below (respectively, above) then $\inf \mathcal{X}(\theta;\psi) = -\infty$ (respectively, $\sup \mathcal{X}(\theta;\psi) = \infty$).

The equivalence of Lehmann-informativeness with the quantile function satisfying the SCP naturally requires an ordering on the set of states Θ. The standard first-order stochastic dominance (FOSD) ordering is sufficient. Because of the centrality of the quantile function to

[4] That is: $\frac{f(x|\theta_2;\kappa)}{f(x|\theta_1;\kappa)}$ is weakly increasing in x if $\theta_2 > \theta_1$, where $f(x|\theta;\kappa)$ denotes the density function corresponding to the distribution function $F(x|\theta;\kappa)$.

the analysis, I include an equivalent formulation of FOSD in terms of the quantile function[5] in the following:

Property 3 *For any regime κ, if $\theta_2 > \theta_1$ then the distribution of X given θ_2 FOSD the distribution of X given θ_1, i.e., $F(x|\theta_2;\kappa) \leq F(x|\theta_1;\kappa)$ for any x, or equivalently, $F^{-1}(1-t|\theta_2;\kappa) \geq F^{-1}(1-t|\theta_1;\kappa)$.*

As noted, Lehmann (1988) and Quah and Strulovici (2009) both impose MLRP, which implies FOSD.[6]

The main result is:

Proposition 2 *Let Properties 1-3 hold. The Lehmann-informativeness of X is increasing in the regime κ if and only if the quantile function $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta \times [0,1]$ has the product ordering.*

The following simple example illustrates Proposition 2.

Example: Let $X = \theta + \frac{\nu}{\kappa}$, where ν is distributed uniformly over $[-1,1]$. Theorem 5.3 in Lehmann establishes that Lehmann informativeness is increasing in κ.[7] The quantile function is $F^{-1}(1-t|\theta;\kappa) = \theta + \frac{1-2t}{\kappa}$. Hence $F^{-1}(1-t_2|\theta_2;\kappa_1) \geq (>) F^{-1}(1-t_1|\theta_1;\kappa_1)$ is equivalent to $\theta_2 - \theta_1 \geq (>) \frac{2t_2-2t_1}{\kappa_1}$; and so if additionally $t_2 \geq t_1$, $\theta_2 \geq \theta_1$, and $\kappa_2 \geq \kappa_1$, then $F^{-1}(1-t_2|\theta_2;\kappa_2) \geq (>) F^{-1}(1-t_1|\theta_1;\kappa_2)$. Hence $F^{-1}(1-t|\theta;\kappa)$ indeed satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta \times [0,1]$ has the product ordering.

1.3 Spence-Mirrlees single-crossing

To check whether the quantile function $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP, it is useful to relate it to the Spence-Mirrlees single-crossing condition, which is expressed in terms of derivatives. Milgrom and Shannon's (1994) Theorem 3 establishes the equivalence (under certain conditions) between the Spence-Mirrlees condition and the SCP under the lexicographic ordering. Under Property 1, $F^{-1}(1-t|\theta;\kappa)$ is strictly decreasing in t, and under Property 3, $F^{-1}(1-t|\theta;\kappa)$ is weakly increasing in θ. Under these conditions, it is straightforward to show that the SCP under the product ordering coincides with the SCP under the lexicographic ordering, which in turn coincides with the Spence-Mirrlees condition.

[5] See, for example, Theorem 4.1 in Levy (1998).

[6] See, for example, Gollier (2001).

[7] Evaluating, $I(x,\theta;\kappa_1,\kappa_2) = \frac{\kappa_2}{\kappa_1}(x-\theta) + \theta$ (regardless of the distribution of ν).

Proposition 3 *Let Properties 1-3 hold.*

(I) $F^{-1}(1-t|\theta;\kappa)$ *satisfies the SCP in* $((\theta,t);\kappa)$*, where* $\Theta\times[0,1]$ *has the product ordering, if and only if* $F^{-1}(1-t|\theta;\kappa)$ *satisfies the SCP in* $((\theta,t);\kappa)$*, where* $\Theta\times[0,1]$ *has the lexicographic ordering.*

(II) If, moreover, $F^{-1}(1-t|\theta;\kappa)$ *is differentiable with respect to* θ *and* t*, with derivatives continuous in* (θ,t,κ)*, then the Lehmann-informativeness of* X *is increasing in the regime* κ *if and only if* $F^{-1}(1-t|\theta;\kappa)$ *satisfies the Spence-Mirrlees single-crossing condition condition, i.e.,*

$$\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|} \text{ is increasing in } \kappa. \tag{2}$$

1.4 A geometric interpretation of Lehmann's ordering

Propositions 2 and 3 establish the equivalence of Lehmann's ordering and single-crossing of the quantile function. Section 2 below illustrates how this equivalence aids the application of Lehmann ordering. In addition, this equivalence leads to the following geometric interpretation of Lehmann's ordering.

Graphically, the SCP corresponds to the isoquants of the quantile function $F^{-1}(1-t|\theta;\kappa)$ in (θ,t) growing steeper as κ increases, as illustrated in Figure 1. In the example above, the isoquants are the lines $t=\frac{1}{2}(1+\kappa\theta-\kappa x)$. Intuitively, steeper isoquants correspond to greater information content, as follows. The observation of a realization of X conveys the same information as the observation of what quantile the realization belongs to. Steeper isoquants correspond to the quantile containing more information about θ, and correspondingly less information about the "noise" term t.

2 Applications

2.1 Learning from the actions of others

In many cases, economic agents learn from the actions of others. For example, one economic agent—the decisionmaker, in the formalism of this paper—may learn about the quality of an investment project by observing a second economic agent's willingness to invest funds in a similar project, where the second economic agent knows θ.

Specifically, suppose that the decisionmaker observes an investment x made by an investor, where the investor chooses the investment x to solve

$$\max_{x\geq 0}\,(1-\psi(\theta,\kappa))\,\bar{u}(x;t)+\psi(\theta,\kappa)\,\underline{u}(x;t)\,, \tag{3}$$

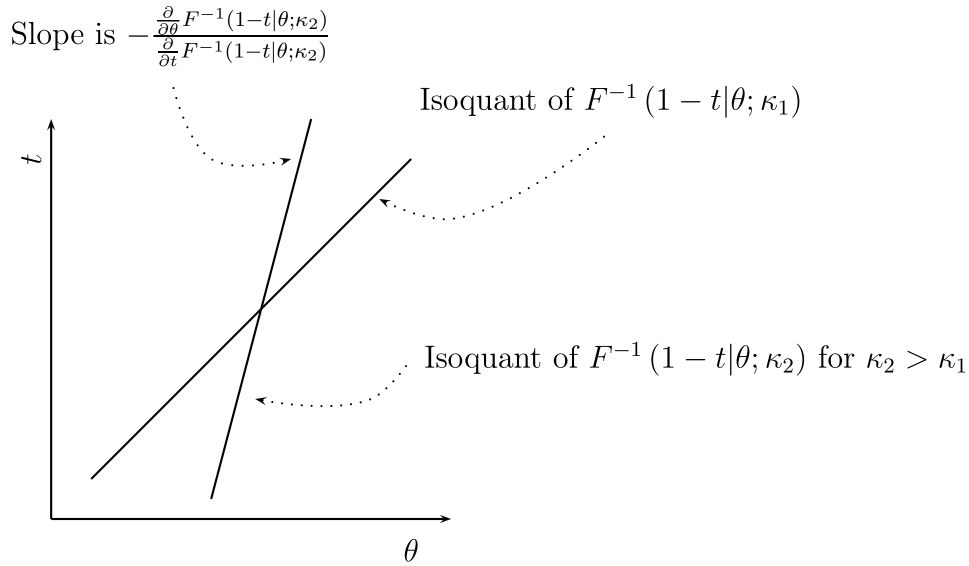

Figure 1: Graphical illustration of Lehmann-informativeness and SCP of the quantile function. Steeper isoquants for κ_2 than for κ_1 correspond to greater Lehmann-informativeness.

where $\bar{u}$ and $\underline{u}$ are both concave in x, with $\bar{u}_x > 0 > \underline{u}_x$. That is, there are "good" and "bad" outcomes, where the bad outcome occurs with probability $\psi(\theta, \kappa)$, and depends on the combination of a state variable θ and a regime κ. For example, κ could be a macroeconomic state, or a government policy. The investment x is beneficial conditional on the good outcome, but costly conditional on the bad outcome. Finally, t represents idiosyncratic factors that affect the investor's willingness to invest, such as the investor's wealth level, or the investor's exposure to other risks. Assume that $t \sim U(0, 1)$. From the perspective of the decisionmaker, t introduces noise into the investor's decision.

Note that "investment" and the "investor" can be broadly interpreted in a number of different ways.

It is often suggested that investors pay more attention to fundamentals—here, θ—if they are more exposed to the risk of bad outcomes. Below, I formalize this idea by deriving a simple condition for when greater exposure to the risk of bad outcomes increases Lehmann-informativeness. In what is close to a normalization, I parameterize ψ so that $\psi_\kappa > 0$ and $\psi_\theta < 0$, i.e., higher values of κ correspond to a higher probability of a bad outcome, and higher values of θ correspond to better fundamentals (lower probability of a bad outcome). A simple example that fits within this general framework is that the bad outcome occurs only if both a project specific event occurs (probability $1 - \theta$) *and* an external event occurs (probability κ), so that

$$\psi = (1 - \theta)\,\kappa. \tag{4}$$

The concavity assumptions on $\bar{u}$ and $\underline{u}$ imply

$$\frac{\partial}{\partial x} \ln \left(-\frac{\bar{u}_x (x;t)}{\underline{u}_x (x;t)} \right) < 0. \tag{5}$$

In addition, suppose that, for all values of θ, t, κ,

$$(1 - \psi(\theta, \kappa)) \bar{u}_x (0;t) + \psi(\theta, \kappa) \underline{u}_x (0;t) > 0,$$

so that investment is always positive; and that the idiosyncratic factor t affects the utility ratio $-\frac{\bar{u}_x}{\underline{u}_x}$ according to

$$\frac{\partial}{\partial t} \ln \left(-\frac{\bar{u}_x (x;t)}{\underline{u}_x (x;t)} \right) \quad < \quad 0, \tag{6}$$

$$\frac{\partial^2}{\partial t \partial x} \ln \left(-\frac{\bar{u}_x (x;t)}{\underline{u}_x (x;t)} \right) \quad \leq \quad 0. \tag{7}$$

To interpret (6), suppose that t is an underlying characteristic that raises the marginal utilities $\bar{u}_x$ and $-\underline{u}_x$. Then (6) says that t raises marginal utility proportionally more in the bad state than in the good state. This is a natural property. For example, it is satisfied in the following two specifications, which correspond to the idiosyncratic term t representing, respectively, the investor's wealth level and risk exposure: (I) $\bar{u}(x;t) = u\left(W(t) + \bar{R}x\right)$ and $\underline{u}(x;t) = u(W(t) + \underline{R}x)$, (II) $\bar{u}(x;t) = (1-t) u\left(\bar{W} + \bar{R}x\right) + tu\left(\underline{W} + \bar{R}x\right)$ and $\underline{u}(x;t) = (1-t) u\left(\bar{W} + \underline{R}x\right) + tu(\underline{W} + \underline{R}x)$, where in both specifications $\underline{R} < 0 < \bar{R}$, and u features decreasing absolute risk aversion (DARA); in (I), $W'(t) < 0$, and in (II), $\underline{W} < \bar{W}$.[8]

Inequality (7) is a regularity condition, and says that the extent to which t raises marginal utility proportionally more in the bad state than in the good state increases as the investor invests more. It is satisfied by both specifications (I) and (II).

Let $x(\theta, t, \kappa)$ denote the investor's optimal investment. Given concavity, it is determined by the first-order condition (FOC) of (3), which can be straightforwardly written as

$$\ln \left(-\frac{\bar{u}_x (x(\theta, t, \kappa); t)}{\underline{u}_x (x(\theta, t, \kappa); t)} \right) + \ln \left(\frac{1 - \psi(\theta, \kappa)}{\psi(\theta, \kappa)} \right) = 0. \tag{8}$$

Differentiation of (8) with respect to θ and t delivers

$$x_\theta (\theta, t, \kappa) \frac{\partial}{\partial x} \ln \left(-\frac{\bar{u}_x (x;t)}{\underline{u}_x (x;t)} \right) \Bigg|_{x = x(\theta, t, \kappa)} \quad = \quad -\frac{\partial}{\partial \theta} \ln \left(\frac{1 - \psi(\theta, \kappa)}{\psi(\theta, \kappa)} \right)$$

[8]The online appendix contains a proof that both specifications (I) and (II) satisfy both (6) and (7).

$$x_t(\theta,t,\kappa)\,\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Big|_{x=x(\theta,t,\kappa)} = -\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Big|_{x=x(\theta,t,\kappa)}$$
$$x_\kappa(\theta,t,\kappa)\,\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Big|_{x=x(\theta,t,\kappa)} = -\frac{\partial}{\partial \kappa}\ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right).$$

From (5), it follows that $x_\theta > 0$, and, from (6), that $x_t < 0$. So Properties 1-3 are satisfied, and the quantile function of X is given by

$$F^{-1}(1-t|\theta;\kappa) = x(\theta,t,\kappa). \tag{9}$$

Moreover, the Spence-Mirrlees ratio is

$$\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|} = -\frac{x_\theta(\theta,t,\kappa)}{x_t(\theta,t,\kappa)} = \frac{\frac{\partial}{\partial\theta}\ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right)}{-\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Big|_{x=x(\theta,t,\kappa)}}. \tag{10}$$

By Proposition 3, the Lehmann-informativeness of the investment x is increasing in the regime κ if and only if the ratio (10) is increasing in κ. Importantly, this condition can be evaluated without solving for the action x.

Recall that $\psi_\kappa > 0$, i.e., as κ increases, the investor is more exposed to the risk of bad outcomes. Hence $x_\kappa < 0$, and so by (7) the denominator in (10) is decreasing in κ.

The ratio $\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}$ is the likelihood ratio of good and bad outcomes for the investor. So by Proposition 3, a sufficient condition for the Lehmann-informativeness of the investment x to increase in exposure to risk (κ) is that the likelihood ratio $\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}$ be log supermodular. To interpret this condition, recall that the ratio $\frac{1-\psi}{\psi}$ is increasing in θ. So log supermodularity says that the likelihood ratio $\frac{1-\psi}{\psi}$ becomes more sensitive to θ as κ increases.

Note that specification (4) satisfies the log-supermodularity condition,[9] and hence generates investment decisions that are more Lehmann-informative in cases with more exposure to risk (higher κ).

2.2 Learning from prices

Instead of a decisionmaker learning from the actions of others, as in the previous subsection, I consider now the case of a decisionmaker who learns from prices. To give a specific example, a lender may seek to learn the riskiness of mortgage lending from the traded prices of mortgage-backed securities. Bond, Edmans, and Goldstein (2012) survey the literature on agents learning from financial prices.

Since the price is the object the decisionmaker is learning from, I denote the price by

[9] That is: $\frac{\partial}{\partial\theta}\ln\frac{1-\psi}{\psi} = \frac{\kappa}{1-(1-\theta)\kappa} + \frac{1}{1-\theta}$, which is increasing in κ.

x. The price is determined by the standard market clearing condition. Specifically, the market is populated by a mixture of informed agents, who observe θ and choose a quantity q to maximize utility $U(q, x, \theta, \kappa)$, and other agents, who trade for idiosyncratic reasons unrelated to either the price x or the state θ. These agents are analogous to the "noise" or "liquidity" traders in Grossman and Stiglitz (1980) and a large subsequent literature. Let the excess demand stemming from these noise traders be $-s(t)$, where $t \sim U(0,1)$ and $s'(t) > 0$. Moreover, I assume that $s(0)$ and $s(1)$ are both finite, and focus on the case in which $s(0)$ and $s(1)$ are sufficiently small, as explained below. The utility function of informed agents takes the form

$$U(q, x, \theta, \kappa) = (1 - \psi(\theta, \kappa))\, u\left(q\left(\bar{R} - x\right)\right) + \psi(\theta, \kappa)\, u\left(q\left(\underline{R} - x\right)\right),$$

where u is increasing and concave and exhibits DARA, $\bar{R} > \underline{R} \geq 0$, and ψ is again parameterized so that $\psi_\kappa > 0$ and $\psi_\theta < 0$. Note that this specification falls outside the CARA-normal framework that is used in many noisy rational expectation models.[10]

An informed agent's demand at price x is $q(x, \theta, \kappa)$, determined by the FOC

$$U_q(q(x, \theta, \kappa), x, \theta, \kappa) = 0. \tag{11}$$

The equilibrium price $x(\theta, t, \kappa)$ is then determined by the market clearing condition

$$q(x(\theta, t, \kappa), \theta, \kappa) = s(t). \tag{12}$$

Assume that $\psi \in (0,1)$ for all values of θ and κ. Then an immediate consequence of market clearing (12) is that the price x lies in the open interval $\left(\underline{R}, \bar{R}\right)$. Hence $U_{q\theta} > 0$ and $U_{q\kappa} < 0$. Moreover, concavity of u implies that $U_{qq} < 0$. It follows straightforwardly that, as one would expect, demand is increasing in θ and decreasing in κ, i.e., $q_\theta > 0$, and $q_\kappa < 0$. Moreover, in the online appendix I establish that demand is decreasing in price, $q_x < 0$.

Differentiation of the market clearing (12) condition yields

$$\begin{aligned} x_\theta q_x + q_\theta &= 0 \\ x_t q_x - s'(t) &= 0 \\ x_\kappa q_x + q_\kappa &= 0. \end{aligned}$$

So $x_\theta > 0$ and $x_t < 0$, implying that the quantile function simply equals the price, i.e., (9)

[10]Breon-Drish (2015) and subsequent papers analyze noisy rational expecation equilibria outside the CARA-normal framework. A key insight in these papers is that one can use the market clearing condition to implicitly characterize equilibrium prices; and the analysis below similarly exploits this observation.

holds, and that Properties 1-3 are satisfied. Moreover, the Spence-Mirrlees ratio is

$$\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|} = -\frac{x_\theta}{x_t} = \frac{q_\theta}{s'(t)}.$$

Hence, by Proposition 3, to evaluate Lehmann-informativeness one must sign

$$\frac{\partial}{\partial\kappa}\left(\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|}\right) = \frac{q_{\theta\kappa}+q_{\theta x}x_\kappa}{s'(t)} = \frac{q_{\theta\kappa}-\frac{q_{\theta x}q_\kappa}{q_x}}{s'(t)} = -\frac{q_\kappa}{s'(t)}\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\Big|_{x=x(\theta,t,\kappa)}. \quad (13)$$

Expression (13) states the Spence-Mirrlees condition in terms of the demand function q, and so can be checked without solving explicitly for prices x. Moreover, further straightforward substitution relates the properties of demand directly to the utility function U: see online appendix for details, both here and below. In particular, for $s(0)$ and $s(1)$ sufficiently small,

$$\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\Big|_{x=x(\theta,t,\kappa)} > \frac{\psi_\theta}{\psi} - \frac{\psi_{\theta\kappa}}{\psi_\kappa}. \quad (14)$$

In words: The RHS of (14) arises from $q_{\theta\kappa}$, i.e., it captures the interaction of θ and κ in determining demand q. Intuitively, this is the key determinant of whether the information content of the price is increasing in κ. The assumption that $s(0)$ and $s(1)$ are sufficiently small ensures that the interaction of θ and the price x in determining demand, $q_{\theta x}$, is of second-order importance.

Consequently, (14) implies that the Lehmann-informativeness of prices is increasing in κ if $\frac{\psi_\theta}{\psi} - \frac{\psi_{\theta\kappa}}{\psi_\kappa} \geq 0$ (recall that $q_\kappa < 0$, $s'(t) > 0$), or equivalently, if the probability ψ of the low realization $\underline{R}$ is weakly log-submodular in (θ,κ). It is immediate that specification (4) satisfies this condition. More generally, log-submodularity holds if the bad-outcome probability becomes more sensitive to the state θ as κ increases.

2.3 Trading constraints and the informational content of prices

I next use the tools from Section 1 to show how constraints on short-sales of assets can lead to prices that are less informative.[11]

For maximal transparency, I adopt a very simple model. Informed investors know θ, have mean-variance preferences, and trade the asset. As before, $q(x,\theta,\kappa)$ denotes informed

[11] Various authors have argued that the combination of short-sales constraints and investor disagreement (stemming, for example, from heterogeneous priors) prevent asset prices impounding negative information (see, e.g., Miller (1977), Hong and Stein (2007)), and leads to inflated asset prices. Here, I focus on how short-sales constraints affect price-informativeness rather than the price-level; and on the effect of short-sales constraints absent disagreement.

agent demand given asset price x, state θ, and regime κ. The regime κ affects shorting costs: positions $q < 0$ incur a per-unit cost of $K(\kappa)$, where without loss K is a decreasing function of the regime κ. So under mean-variance preferences, informed agent demand takes the simple form

$$q(x, \theta, K) = \begin{cases} \theta - x & \theta > x \\ 0 & \theta \in [x - K, x] \\ \theta - x + K & \theta < x - K \end{cases}, \tag{15}$$

where for expositional transparency I have normalized the residual variance and risk aversion parameters so that there is no further multiplicative constant. As is widely appreciated, transaction costs lead to a kinked demand function, with an interval of prices over which, for a given θ, informed agents do not take either long or short positions in the asset, i.e., have locally perfectly inelastic demand.

There are also "liquidity" sellers who trade in response to urgent consumption needs, and sell all their shares, regardless of price. The number $\lambda(t)$ of such traders is uncertain, and determined by the realization of $t \in (0, 1)$. The function λ is strictly positive for all realizations of t, and is strictly increasing in t.

In addition, there is a further measure A of buyers who receive a wealth endowment, and spend all this endowment buying shares, resulting in demand $\frac{A}{x}$.[12]

The market-clearing condition is consequently

$$q(x, \theta, K) - \lambda(t) + \frac{A}{x} = 0. \tag{16}$$

The LHS of (16) is strictly decreasing in the price x, and is strictly positive as $x \to 0$ and strictly negative as $x \to \infty$. Hence a positive-valued solution exists x, and is unique.

In contrast to the applications to subsections 2.1 and 2.2, the equilibrium value of x (here, price) can be solved for analytically, and takes the following simple form (here and below, all details are relegated to the online appendix). First, define

$$\xi(\theta, t) = \frac{1}{2}\left(\theta - \lambda(t) + \sqrt{(\theta - \lambda(t))^2 + 4A}\right).$$

Then the equilibrium price is

$$x(\theta, t, K) = \begin{cases} \xi(\theta, t) & \text{if } \theta > \frac{A}{\lambda(t)} \\ \frac{A}{\lambda(t)} & \text{if } \theta \in \left[\frac{A}{\lambda(t)} - K, \frac{A}{\lambda(t)}\right] \\ \xi(\theta + K, t) & \text{if } \theta < \frac{A}{\lambda(t)} - K \end{cases}. \tag{17}$$

[12]The presence of liquidity "buyers" is needed to ensure that informed traders sometimes take short positions; if informed traders never take short positions then shorting costs are irrelevant.

Note that $\xi\left(\frac{A}{\lambda(t)}, t\right) = \frac{A}{\lambda(t)}$, so that the price x is continuous in (θ, t). As one would expect, shorting costs generate an interval of fundamentals over which the price is independent of the fundamental.

Note that $x(\theta, t, K)$ is increasing in θ, which ensures that Properties 1-3 are satisfied; and is strictly decreasing in t, so that the quantile function simply equals the price, i.e., (9) holds.

The price (17) does not satisfy the differentiability requirements needed to apply Proposition 3. But it is nonetheless relatively straightforward to show that the price x satisfies SCP in $((\theta, t); \kappa)$. By Proposition 2, it follows that Lehmann-informativeness is increasing in κ, i.e., is decreasing in the shorting cost K.

2.4 Inflation and the information content of prices

As Ball and Romer (2003) observe, economists often argue that "inflation reduces the efficiency of the price system." Ball and Romer consider a model in which firms set prices every two periods, and consumers seek to learn about the expected real price of each firm's product. Here, I recast this idea in a continuous-time setting with an arbitrary frequency of price-resetting, and relate the informational content of prices to Lehmann.

Let δ denote the inflation rate. The length of time between each firm's (nominal) price changes is s. Let P_H be the price the firm selects when it has the opportunity to change prices. Consequently, the expected real price of the firm's product is $\bar{P} \equiv P_H \int_0^s e^{-\delta\tilde{t}} d\tilde{t}$.

As in Ball and Romer, a consumer observes a product's real price P at an instant in time, but does not know the time t that has elapsed since the last price-change. The consumer wishes to infer the expected real price $\bar{P}$.

Hence a consumer observes

$$P = P_H e^{-\delta t} = \frac{e^{-\delta t}}{\int_0^s e^{-\delta\tilde{t}} d\tilde{t}} \bar{P},$$

and wishes to infer $\bar{P}$.

Taking logs and disregarding constant terms, the consumer effectively observes $\ln \bar{P} - \delta t$, and wants to infer $\ln \bar{P}$. Because the consumers don't know the last price-change date, the time-since-price-change t is uniformly distributed over $[0, s]$ from their perspective. This is exactly the example presented following Proposition 2. As such, a reduction in inflation δ is associated with an increase in the Lehmann-informativeness of prices, but isn't associated with an increase in Blackwell informativeness.

2.5 Measuring information content with conditional variance

Subsections 2.2 and 2.3 analyze the information content of financial prices. In many studies of financial markets, the information content of prices is measured by their ability to predict future cash flows, as measured using second moments. In the notation of this paper, this boils down to the conditional variance $var(\theta|X)$, i.e., θ is a future cash flow, X is the current price, and greater residual variance corresponds to lower informativeness.[13]

Lehmann-informativeness implies the conditional variance ordering, as follows. Suppose that X_2 is more Lehmann-informative than X_1. By Theorem 1(i) of Ganuza and Penalva (2010), it follows that $E\left[E[\theta|X_2]^2\right] \geq E\left[E[\theta|X_1]^2\right]$. By the law of total expectation, $E[E[\theta|X_2]] = E[E[\theta|X_1]] = E[\theta]$. Consequently, $var(E[\theta|X_2]) \geq var(E[\theta|X_1])$. By the law of total variance, it follows that $E[var(\theta|X_2)] \leq E[var(\theta|X_1)]$, i.e., X_2 is more informative under the conditional variance ordering.

3 Conclusion

Lehmann's (1988) information ordering is equivalent to a single-crossing property of the quantile function. Under mild differentiability conditions, Lehmann's ordering is also equivalent to Spence-Mirrlees single-crossing. These equivalences, which have not previously been noted, considerably aid the application of Lehmann's ordering.

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[13]See, for example, Brunnermeier (2005), Peress (2010), Bai et al (2016). Closely related are Dávila and Parlatore (2018), who define price informativeness using $var(X|\theta)$, i.e., the residual variance of current prices conditional on future cash flow innovations.

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A Proofs of Propositions 2 and 3

Proof of Proposition 2:

SCP implies Lehmann-informativeness: Fix κ_1, $\kappa_2 > \kappa_1$, $x \in \mathcal{X}(\kappa_2)$ and $\theta_1, \theta_2 \in \Theta(x; \kappa_2)$ with $\theta_2 \geq \theta_1$. Let t_1 and t_2 be such that

$$F^{-1}(1 - t_1|\theta_1; \kappa_2) = F^{-1}(1 - t_2|\theta_2; \kappa_2) = x. \tag{18}$$

Hence

$$F(x|\theta_1; \kappa_2) = 1 - t_1 \tag{19}$$
$$F(x|\theta_2; \kappa_2) = 1 - t_2. \tag{20}$$

By FOSD, it follows that $t_2 \geq t_1$. Equations (19) and (20) also deliver

$$\begin{aligned} F\left(F^{-1}(1 - t_1|\theta_1; \kappa_1)|\theta_1; \kappa_1\right) &= 1 - t_1 = F(x|\theta_1; \kappa_2) \\ F\left(F^{-1}(1 - t_2|\theta_2; \kappa_1)|\theta_2; \kappa_1\right) &= 1 - t_2 = F(x|\theta_2; \kappa_2). \end{aligned}$$

Hence

$$\begin{aligned} I(x, \theta_1; \kappa_1, \kappa_2) &= F^{-1}(1 - t_1|\theta_1; \kappa_1) \\ I(x, \theta_2; \kappa_1, \kappa_2) &= F^{-1}(1 - t_2|\theta_2; \kappa_1). \end{aligned}$$

Equality (18) and the SCP then imply

$$F^{-1}(1 - t_2|\theta_2; \kappa_1) \leq F^{-1}(1 - t_1|\theta_1; \kappa_1),$$

establishing the result.

Lehmann-informativeness implies SCP:

Suppose that, contrary to the claimed result, the SCP is violated, i.e., there exist t_1, $t_2 \geq t_1$, θ_1, $\theta_2 \geq \theta_1$, κ_1 and $\kappa_2 \geq \kappa_1$ such that either

$$F^{-1}(1 - t_2|\theta_2; \kappa_1) = F^{-1}(1 - t_1|\theta_1; \kappa_1) \tag{21}$$
$$F^{-1}(1 - t_2|\theta_2; \kappa_2) < F^{-1}(1 - t_1|\theta_1; \kappa_2), \tag{22}$$

or

$$F^{-1}(1 - t_2|\theta_2; \kappa_1) > F^{-1}(1 - t_1|\theta_1; \kappa_1) \tag{23}$$

$$F^{-1}(1-t_2|\theta_2;\kappa_2) \quad \leq \quad F^{-1}(1-t_1|\theta_1;\kappa_2). \tag{24}$$

The first step is to find t_3 such that

$$F^{-1}(1-t_3|\theta_2;\kappa_1) \quad > \quad F^{-1}(1-t_1|\theta_1;\kappa_1) \tag{25}$$

$$F^{-1}(1-t_3|\theta_2;\kappa_2) \quad = \quad F^{-1}(1-t_1|\theta_1;\kappa_2). \tag{26}$$

There are two subcases. First, if (23) and (24) hold, with (24) at equality, then simply set $t_3 = t_2$. Second, if instead $F^{-1}(1-t_2|\theta_2;\kappa_2) < F^{-1}(1-t_1|\theta_1;\kappa_2)$ then by Property 3,

$$F^{-1}(1-t_2|\theta_2;\kappa_2) < F^{-1}(1-t_1|\theta_1;\kappa_2) \leq F^{-1}(1-t_1|\theta_2;\kappa_2).$$

So by Property 1, there exists t_3 such that $t_1 \leq t_3 < t_2$ satisfying (26). Moreover, since $t_3 < t_2$ and $F^{-1}(1-t_2|\theta_2;\kappa_1) \geq F^{-1}(1-t_1|\theta_1;\kappa_1)$ it follows that (25) holds.

Let $x = F^{-1}(1-t_1|\theta_1;\kappa_2) \in \mathcal{X}(\theta_1;\kappa_2)$. From (26) and the definition of x

$$F(x|\theta_1;\kappa_2) \quad = \quad 1-t_1$$

$$F(x|\theta_2;\kappa_2) \quad = \quad 1-t_3.$$

By an identical argument to that used in the first half of the proof,

$$I(x,\theta_1;\kappa_1,\kappa_2) \quad = \quad F^{-1}(1-t_1|\theta_1;\kappa_1)$$

$$I(x,\theta_2;\kappa_1,\kappa_2) \quad = \quad F^{-1}(1-t_3|\theta_2;\kappa_1).$$

So the Lehmann-informativeness condition implies

$$F^{-1}(1-t_3|\theta_2;\kappa_1) \leq F^{-1}(1-t_1|\theta_1;\kappa_1),$$

contradicting (25) and completing the proof.

Proof of Proposition 3:

Part (I): If (θ_2, t_2) exceeds (θ_1, t_1) under the product order, it does so under the lexicographic order also. As such, it is immediate that if $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP under the lexicographic order, it does so under the product order also. To establish the opposite implication, consider (θ_1, t_1), (θ_2, t_2), κ_1, and κ_2 such that (θ_2, t_2) exceeds (θ_1, t_1) under the lexicographic order; $\kappa_2 > \kappa_1$; and $F^{-1}(1-t_2|\theta_2;\kappa_1) \geq F^{-1}(1-t_1|\theta_1;\kappa_1)$. The only nontrivial case to consider is that in which (θ_2, t_2) *does not exceed* (θ_1, t_1) under the product order, i.e., $\theta_2 \geq \theta_1$ but $t_2 < t_1$. In this case, Properties 1 and 3 imply that, for any κ,

$$F^{-1}(1 - t_2|\theta_2; \kappa) \geq F^{-1}(1 - t_2|\theta_1; \kappa) > F^{-1}(1 - t_1|\theta_1; \kappa),$$

completing the proof.

Part (II): As noted in the main text, Part (II) is an application of Milgrom and Shannon's (1994) Theorem 3. To apply this result it is necessary to verify the condition that $F^{-1}(1 - t|\theta; \kappa)$ is completely regular, which, given that F^{-1} is weakly increasing in θ, is equivalent to checking that if

$$F^{-1}(1 - t_1|\theta_1; \kappa) = F^{-1}(1 - t_2|\theta_2; \kappa)$$

for some $\theta_2 > \theta_1$, then for any $\theta \in (\theta_1, \theta_2)$ there exists $t(\theta)$ continuous in θ such that

$$F^{-1}(1 - t(\theta)|\theta; \kappa) = F^{-1}(1 - t_1|\theta_1; \kappa). \tag{27}$$

This condition is indeed satisfied since, by Property 3,

$$F^{-1}(1 - t_1|\theta; \kappa) \geq F^{-1}(1 - t_1|\theta_1; \kappa) = F^{-1}(1 - t_2|\theta_2; \kappa) \geq F^{-1}(1 - t_2|\theta; \kappa),$$

and hence (by Property 1) there exists a unique $t(\theta)$ such (27) holds. Continuity follows since F^{-1} is continuous in (θ, t, κ).

B Proof of Proposition 1

The heart of proof of Proposition 1 is the following result, which generalizes Step 2 of Lemma 3 in Quah and Strulovici (2009) to the case in which the action space B is non-compact.

Lemma 1 *If $b(\theta)$ is a weakly decreasing function then there exists b^* such that $V(b^*, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta$.*

Proof of Lemma 1: Consider first the case in which $b(\cdot)$ takes only finitely many values. Hence there is finite partition $\{\Theta_k : k = 1, \ldots, K\}$ of Θ such that $b(\cdot)$ is constant over each partition element Θ_k, and every member of Θ_{k+1} exceeds every member of Θ_k. The proof establishes the slightly stronger result that there exists $b^* \in [b(\Theta_K), b(\Theta_1)]$ such that $V(b^*, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta$.

The proof is by induction. Suppose there exists $\tilde{b}_k \geq b(\Theta_k)$ such that $V\left(\tilde{b}_k, \theta\right) \geq V(b(\theta), \theta)$ for all $\theta \in \bigcup_{j \leq k} \Theta_j$. To establish the result, it is sufficient to establish the inductive step that there exists $\tilde{b}_{k+1} \geq b(\Theta_{k+1})$ such that $V\left(\tilde{b}_{k+1}, \theta\right) \geq V(b(\theta), \theta)$ for all $\theta \in \bigcup_{j \leq k+1} \Theta_j$. Define $\tilde{b}_{k+1}$ as the supremum of

$$\arg \max_{b \in \left[b(\Theta_{k+1}), \tilde{b}_k\right]} V(b, \sup \Theta_k).$$

So in particular, $V\left(\tilde{b}_{k+1}, \sup \Theta_k\right) \geq V(b(\Theta_{k+1}), \sup \Theta_k)$. Since b is constant over Θ_{k+1}, SCP implies $V\left(\tilde{b}_{k+1}, \theta\right) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta_{k+1}$. Moreover, $V\left(\tilde{b}_{k+1}, \theta\right) \geq V\left(\tilde{b}_k, \theta\right)$ for all $\theta \in \bigcup_{j \leq k} \Theta_j$, since if instead $V\left(\tilde{b}_k, \theta\right) > V\left(\tilde{b}_{k+1}, \theta\right)$ for some $\theta \in \bigcup_{j \leq k} \Theta_j$, SCP implies that $V\left(\tilde{b}_k, \sup \Theta_k\right) > V\left(\tilde{b}_{k+1}, \sup \Theta_k\right)$, which contradicts the definition of $\tilde{b}_{k+1}$. By supposition, it then follows that $V\left(\tilde{b}_{k+1}, \theta\right) \geq V(b(\theta), \theta)$ for all $\theta \in \bigcup_{j \leq k+1} \Theta_j$, establishing the inductive step and hence completing the proof of this case.

Next, consider the case in which $b(\cdot)$ take infinitely many values. Recall that $\underline{\theta}$, $\bar{\theta}$, $\underline{b}$ and $\bar{b}$ are defined in subsection 1.1. Define

$$\beta(\theta) = \begin{cases} \min\left\{b(\theta), \max\left\{b(\underline{\theta}), \bar{b}\right\}\right\} & \text{if } \theta \leq \underline{\theta} \\ b(\theta) & \text{if } \theta \in \left(\underline{\theta}, \bar{\theta}\right) \\ \max\left\{b(\theta), \min\left\{b\left(\bar{\theta}\right), \underline{b}\right\}\right\} & \text{if } \theta \geq \bar{\theta} \end{cases}.$$

Define $\bar{B} = \left[\min\left\{b\left(\bar{\theta}\right), \underline{b}\right\}, \max\left\{b(\underline{\theta}), \bar{b}\right\}\right]$. Observe that β is weakly decreasing and $\beta(\Theta) \subset \bar{B}$. Moreover, if $\beta(\theta) \neq b(\theta)$ then either $\theta \leq \underline{\theta}$ and $b(\theta) > \beta(\theta) \geq \bar{b}$, or $\theta \geq \bar{\theta}$ and $b(\theta) < \beta(\theta) \leq \underline{b}$. So by the definition of $\underline{\theta}$, $\bar{\theta}$, $\underline{b}$ and $\bar{b}$,

$$V(\beta(\theta), \theta) \geq V(b(\theta), \theta) \text{ for all } \theta \in \Theta. \tag{28}$$

Let $\{B_n\}$ be a sequence of finite subsets of $\bar{B}$ such that $B_n \subset B_{n+1}$ and $\bigcup_n B_n$ is dense in $\bar{B}$. Define $\beta_n(\theta)$ as the largest member of B_n that is weakly less than $\beta(\theta)$. Hence for any $\theta \in \Theta$, $\beta_{n+1}(\theta) \geq \beta_n(\theta)$ and $\beta_n(\theta) \to \beta(\theta)$.

For any n, the first part of the proof implies that there exists b_n^* such that $V(b_n^*, \theta) \geq V(\beta_n(\theta), \theta)$ for all $\theta \in \Theta$. Moreover, $b_n^* \in \bar{B}$. Hence b_n^* has a convergent subsequence, with limit b^*. By the continuity of V in its first argument, it follows that $V(b^*, \theta) \geq V(\beta(\theta), \theta)$ for all $\theta \in \Theta$. The result then follows from (28), completing the proof.

Proof of Proposition 1: Under Property 1, for any θ, $I(\cdot, \theta)$ is strictly increasing. Let $J(\cdot, \theta) : \mathcal{X}(\theta; \kappa_1) \to \mathcal{X}(\theta; \kappa_2)$ be the inverse of $I(\cdot, \theta)$ with respect to its first argument. Note that $J(\cdot; \theta)$ is strictly increasing.

Note first that X in regime κ_2 and $J(X, \theta)$ in regime κ_1 have the same distribution, since for any $x \in \mathcal{X}(\theta;\kappa_2)$,

$$\begin{aligned}
\Pr\left(X \leq x|\theta;\kappa_2\right) &= F\left(x|\theta;\kappa_2\right) \\
&= F\left(I\left(x,\theta\right)|\theta;\kappa_1\right) \\
&= \Pr\left(X \leq I\left(x,\theta\right)|\theta;\kappa_1\right) \\
&= \Pr\left(J\left(X,\theta\right) \leq J\left(I\left(x,\theta\right),\theta\right)|\theta;\kappa_1\right) \\
&= \Pr\left(J\left(X,\theta\right) \leq x|\theta;\kappa_1\right).
\end{aligned}$$

By the Lehmann-informativeness property, for any $x \in \mathcal{X}(\kappa_2)$ the function $\zeta\left(I\left(x,\theta\right)\right)$ is weakly decreasing in θ over $\Theta\left(x;\kappa_2\right)$. So by Lemma 1, there exists a function $\phi : \mathcal{X}(\kappa_2) \rightarrow B$ such that, for any $x \in \mathcal{X}(\kappa_2)$,

$$V\left(\phi\left(x\right),\theta\right) \geq V\left(\zeta\left(I\left(x,\theta\right)\right),\theta\right) \text{ for all } \theta \in \Theta\left(x;\kappa_2\right).$$

It follows that, for any θ and $\bar{V}$,

$$\begin{aligned}
\Pr\Big(V\left(\phi\left(X\right),\theta\right) \leq \bar{V}|\theta;\kappa_2\Big) &= \Pr\Big(V\left(\phi\left(J\left(X,\theta\right)\right),\theta\right) \leq \bar{V}|\theta;\kappa_1\Big) \\
&\leq \Pr\Big(V\left(\zeta\left(I\left(J\left(X,\theta\right),\theta\right)\right),\theta\right) \leq \bar{V}|\theta;\kappa_1\Big) \\
&= \Pr\Big(V\left(\zeta\left(X\right),\theta\right) \leq \bar{V}|\theta;\kappa_1\Big),
\end{aligned}$$

where the inequality uses $J\left(X,\theta\right) \in \mathcal{X}(\theta;\kappa_2)$, completing the proof.