Share Issues versus Share Repurchases

Philip Bond, Yue Yuan, Hongda Zhong*

December 22, 2023

Abstract

Almost all firms repurchase shares through open market repurchase (OMR) programs. In contrast, issue methods are more diverse: both at-the-market offerings, analogous to OMR programs, and SEOs, analogous to rarely-used tender-offer repurchases, are used by significant fractions of firms. Furthermore, average SEOs are larger than at-the-market offerings. We show that this asymmetry in the diversity of transaction methods in issuances and repurchases and the size-method relation in issuances are natural consequences of the single informational friction of a firm having superior information to investors. Moreover, repurchasing firms are likely maximizing long-term shareholders' payoffs rather than boosting short-term share prices.

*Philip Bond is at the University of Washington. Yue Yuan is at University College London. Hongda Zhong is at the University of Texas at Dallas and CEPR. We thank Sebastian Gryglewicz, Ulrich Hege, Jangwoo Lee, Will Liu, Ernst Maug, John Nash, Sergio Vicente, Anjan Thakor, "Vish" Viswanathan, Hong Zhang and the seminar and conference participants at the London School of Economics, University of Rochester, CICF, SGF Conference, Tsinghua University, Esade Spring Workshop, FTG Summer Meeting, BSE Summer Forum, EFA, Corporate Finance Day, SFS Cavalcade Asia-Pacific, AFA and Renmin University of China for valuable discussions and comments. Yue Yuan gratefully acknowledges the financial support from the Paul Woolley Centre. Hongda Zhong gratefully acknowledges the support of the Economic and Social Research Council [ESRC New Investigator Grant: ES/T003758/1].

1 Introduction

Public firms often tap into the equity market, both issuing new shares to raise funds, and repurchasing existing shares to return cash to investors. In many ways, issuing and repurchasing shares are mirror images of each other. Both types of transaction are subject to informational frictions arising from firms' superior knowledge. And for both types of transaction, firms choose transaction size and method. Conceptually, share repurchases are simply negative issuances.

In this paper, we analyze the two transactions side-by-side, under the assumption that firms have superior knowledge about their own prospects, and can choose both transaction size and method. Although many papers analyze security transactions under asymmetric information, the comparison of issues and repurchases is new to the literature, and yields fresh insights. We emphasize three points.

First, and despite the conceptual symmetry between issue and repurchase transactions, their equilibrium *outcomes* are *not* mirror images of each other. Repurchasing firms cannot signal via the efficiency of transaction method—*money burning*—while issuing firms can. Empirically, almost all firms repurchase via open-market transactions; while issuing firms use both seasoned equity offerings (SEOs) and at-the-market offerings (ATMs) with significant frequencies, though the latter has received limited academic attention. Our analysis rationalizes both patterns.

Second, and in contrast, reducing repurchase sizes is a viable signal for repurchasing firms, just as reducing issue size is a viable signal for issuing firms. The predicted patterns of transaction size and market response are consistent with empirical evidence on both issues and repurchases. The contrast between the first and second points highlights that while reducing repurchase size "burns money" by reducing transaction surplus, doing so also has the separate effect of increasing a firm's total value.

Third, and more conceptually, our analysis isolates a precise formal role for firm value, viz., for any transaction under consideration, a manager should ask, "by what percent will this transaction affect firm value?" The point is starkest for the case of repurchases, in which case transaction size affects transaction surplus and firm value in different directions. But even for issue decisions, a focus on total firm value sheds light on firms' preferences for signalling-via-issue-size over signalling-via-method, and operationalizes Viswanathan's (1995) results on the ordering of signals in terms of standard financial quantities.

Our analysis also speaks to the question of whether firms' capital transactions—and repurchases in particular—are driven by a desire to boost short-term share prices at the expense of long-term shareholder payoffs. Our analysis suggests that they are not. Specifically: if one feeds the assumption that firms heavily weight short-term prices into our analysis then it yields counterfactual implications for repurchase behavior (see Section 5).

Finally, and along similar lines, our analysis speaks to whether a firm's private information is about its assets-in-place or the profitability of "investment" opportunities, a distinction that dates back at least to Myers and Majluf (1984). Our results suggest that firms have private information about assets-in-place; if instead all private information were about investment opportunities, we obtain the counterfactual prediction that a firm's choice of issue and repurchase size is independent of its private information (see Section 6).

In more detail, we model issues and repurchases in a unified and symmetric way. A firm privately knows the value of its assets in place (Myers and Majluf (1984)), and has a surplus-creating "project" that can only be implemented through trading equity. If the project requires a positive investment, the firm needs to raise capital by issuing shares. In contrast, if the "investment" is negative, then the firm needs to pay out capital by repurchasing equity; here, the surplus stems from the avoidance of wasteful expenditures that would take place if cash were instead retained. The project is scalable and produces more surplus if more capital is deployed (raised or paid out) up to some maximum. Firms choose both project size—or equivalently, transaction size—and transaction method associated with different levels of efficiency.

The following two points underpin many of our results. First, while issuing firms want to raise investors' perceptions of their value, repurchasing firms instead want to lower perceptions. Second, equity transactions mechanically affect firm value even without generating surplus. In the textbook case of public information, only transaction surplus matters for firm decisions (i.e., NPV maximization). In contrast, under asymmetric information the total firm value significantly affects firm decisions too. More specifically, an action needs to simultaneously satisfy two conditions to be a viable signal: it decreases surplus, and its effect on the firm value is more favourable to the firms who want to distinguish their types relative to those who want to pool with others (the single crossing property).

Repurchasing firms are unable to signal via money-burning because it is worse firms that want to reveal their types, but money-burning is proportionally more costly for such firms. On the other hand, issuing firms are able to signal via money-burning because it is better firms that want to reveal their types, and money-burning is proportionally cheaper for these firms.

In contrast to this asymmetry, both repurchasing and issuing firms can signal via reducing transaction size. A reduction in repurchases increases firm value, and does so proportionally more for worse firms. A reduction in issuance reduces firm value, but does so proportionally less for better firms.

Our main implications fit well with empirical findings. We start with the "asymmetry" prediction. In principle, similar transaction methods are available for issuing and repurchasing firms. Specifically, firms can raise equity through an SEO in a one-off transaction, which typically completes in 2-8 weeks (Gao and Ritter, 2010); or more smoothly through at-the-market offerings (ATM) over a couple of years.[1] Likewise, repurchases can be carried out either one-off in tender offers (henceforth, TOR, which are often completed within a month (Masulis, 1980)) or smoothly via open market repurchase (OMR) programs that typically last several years (Stephens and Weisbach, 1998). Our asymmetry prediction gives an explanation for the prominent empirical feature that both SEOs and ATMs coexist as frequently observed issue methods, whereas OMR dominates the repurchase market.[2]

Second, the prediction that transaction size reveals firm fundamentals in both issues and repurchases again fits the data well: returns are higher following smaller issues and larger repurchases.[3]

Third, our model's implication that issuing firms prefer to signal via smaller issues rather than via more inefficient methods implies the following pattern: The worst firms issue the maximum amount using the most efficient method; better firms issue less, still using the most efficient method; and the best firms issue the minimum amount possible to fund the project, but use more inefficient issue methods. Consistent with the prediction, empirically the issue method is correlated with issue size. We microfound the efficiency associated with the transaction methods (SEO and ATM when firms issue, OMR and TOR when repurchase) in Section 4. There, we argue that one-off SEOs are more efficient than smoother ATMs, as the former allows the firm to immediately implement the project, whose NPV might diminish over time. Billett, Floros, and Garfinkel (2019) provide evidence that proceeds from an SEO are indeed larger than total proceeds from an average ATM program.[4]

1.1 Related Literature

There is a large literature on firms' capital transaction when they have superior information over investors. When selling securities, costly retention of unsold securities or broadly speaking, transaction size, can be informative signals about firms' hidden quality (see Leland and Pyle (1977), Myers and Majluf (1984), Krasker (1986), and DeMarzo and Duffie (1999)). When repurchasing securities, firms can similarly signal by different repurchase amounts (see Vermaelen (1984), Brennan and Kraus (1987), Ofer and Thakor (1987), Constantinides and Grundy (1989), Chowdhry and Nanda (1994), Lucas and McDonald (1998), and Bond and Zhong (2016)). In general, higher quality

[1] See Billett, Floros, and Garfinkel (2019) for an overview of ATMs, which are growing in popularity.

[2] Billett, Floros, and Garfinkel (2019) document that ATMs represented 63% incidences and 26% issue proceeds of those for SEOs in 2016. In contrast, in 2004, there were 466 cases of OMR with a total size of $223 billion, and tender offers and Dutch auctions only accounted for 18 and 10 cases, and $1.3 billion and $3.9 billion proceeds respectively (see Banyi, Dyl, and Kahle (2008), and similar patterns have been documented by Grullon and Ikenberry (2000)).

[3] On the size-return relation, Asquith and Mullins (1986) and Masulis and Korwar (1986) document negative relation between issue size and announcement return. In tender offer repurchases, Vermaelen (1981) find abnormal return is positively related to target tender fraction.

[4] We calculate from Table 2 of Billett, Floros, and Garfinkel (2019) that the average proceeds per SEO are $256 million, whereas average proceeds per ATM program are $92 million. Even though the ratio of proceeds to market equity is roughly the same between the two methods (18% for SEO and 20% for ATM), it is significantly smaller for ATM than for SEO after controlling for other observable factors (see Table 4 of the same paper).

firms buy more or sell less (or even not sell at all). In addition to transaction-size signaling, these papers also show that firms can signal through tax-inefficient dividend payouts, or more generally burning cash (for example advertisement signaling in Milgrom and Roberts (1986)), in exchange for a more favorable transaction price. Our analysis contributes to this literature by allowing both size and efficiency signaling simultaneously and compares the two directions of equity transactions (issues and repurchases) side by side. Novel to the literature is the insight that firms can use both transaction size and efficiency as signals when they issue, whereas only size signal is possible when repurchase. We also establish that issuing firms prefer to signal via issuing less rather than via issuing inefficiently.

Like us, Babenko, Tserlukevich, and Wan (2020) consider issues and repurchases in a unified model, though from a very different perspective. They show that a firm can profitably trade its own equity (market timing), but in doing so harms shareholders who trade against the more informed firm. In contrast, our paper focuses on how these issues and repurchases are carried out, namely the choices of transaction size and method (efficiency).

Our paper is also related to the literature on firms' choice of equity transaction methods. Brennan and Thakor (1990) and Oded (2011) study firms' choice between tender offer and open market repurchases. In contrast to our model, which studies firms' choice under private information, these papers consider the interaction between informed and uninformed shareholders in their tendering strategies, and emphasize the role of shareholders' endogenous decision to acquire information. In contrast, when firms raise equity, Burkart and Zhong (2023) compare public offerings and rights offerings. The key driver in their paper is the wealth transfer between constrained and unconstrained shareholders, and the efficiency choice is left out of the model. Chemmanur and Fulghieri (1994) present a model in which investment banks endogenously acquire information as underwriters, and predict that firms choose underwritten issues over direct issues unless they face little information asymmetry or receive too low an evaluation from the investment bank to procure its services. In contrast, abstracting from the role of underwriters, we analyze firms' choices between one-off SEOs and smoother ATMs, emphasizing their differences in efficiency in funding corporate investment.

Our paper also speaks to the literature on multi-dimensional signaling/screening. We defer a fuller discussion of this point until page 14 below.

2 The model

We model share issues and repurchases in a unified framework. Consider a firm with assets in place a and an opportunity to invest i in a new project. The value of assets in place, a, is the firm's private information, whereas others only know that a is distributed according to $F(\cdot)$, which admits a density and has support $[a_{\min}, a_{\max}]$. We refer to a as the firm's type.

The firm chooses investment, i.e., project size, i to lie in the closed interval between I_L and I_H, where I_L and I_H are exogenous constants that are common knowledge. Either $I_H > I_L \geq 0$, in which case the project is an investment project; or $I_H < I_L \leq 0$, in which case the project is a divestment project. The case $|I_L| > 0$ corresponds to a minimum project size. This arises naturally for investment projects that have minimum scale. Similarly, a minimum size for a divestment project arises if a firm is compelled to pay out at least a minimum amount of cash; for example, if retaining cash above some level would lead to extremely wasteful spending.

(We note that we have also fully analyzed the case in which a firm always has the option of choosing $i = 0$, i.e., $i \in \{0\} \cup [I_L, I_H]$. In particular, this specification is natural to consider for investment projects. The analysis of this case doesn't yield any additional insights relative to $i \in [I_L, I_H]$, and so we focus on this latter case both for transparency, and in order to preserve symmetry across the analysis of issues and repurchases.) [5]

The investment i is associated with equity transactions: Investment projects ($i > 0$) require funding and hence share issues, while divestment projects ($i < 0$) produce cash to be paid out via repurchases. (For reasons outside the model, the firm prefers to raise funding via equity to other securities, and to pay out cash via repurchases rather than dividends.)[6]

In addition to investment i, the firm can also choose among equity transaction methods with different levels of efficiencies, captured by the variable $\theta \in [0, 1]$, with efficiency increasing in θ. In empirical applications we typically interpret efficiency in terms of whether a transaction occurs at a single point in time, as in SEOs and tender offer repurchases, or smoothly over time, as in at-the-market offerings and open market repurchases. See Section 4 for full details.

Equity transactions are carried out at the competitively determined price $P(i, \theta)$. That is: After a firm announces its investment and transaction efficiency choices (i, θ), competitive investors update their beliefs about the firm type a, and the price $P(i, \theta)$ reflects these updated beliefs.

An equity transaction i carried out with efficiency θ yields surplus $S(i, \theta)$, increasing in transaction size $|i|$ and efficiency θ. While we can easily accommodate more general function forms, for

[5] Effectively, for $I_L > 0$ (the issue setting) we are assuming, in terms of formal objects defined below, that

$$\frac{V(a_{\max}, I_L, 1)}{1 + \frac{I_L}{V(a_{\min}, I_L, 1) - I_L}} > V(a_{\max}, 0, 1),$$

i.e., the best firm prefers issuing I_L at full efficiency but at the most unfavorable price that can be supported in equilibrium over the alternative of doing nothing; along with the analogous assumption for repurchase ($I_L < 0$):

$$\frac{V(a_{\min}, I_L, 1)}{1 + \frac{I_L}{V(a_{\max}, I_L, 1) - I_L}} > V(a_{\min}, 0, 1).$$

[6] For example, in the US dividends are tax-disadvantaged relative to paying out cash through share repurchases, even after the tax changes associated with the 2003 Jobs and Growth Tax Relief Reconciliation Act; see, for example Chetty and Saez (2005) and Blouin, Raedy, and Shackelford (2011).

transparency we parameterize S by

$$S(i,\theta) = i\theta b,$$

where b is a constant that parameterizes surplus created by the investment. For investment projects ($I_H \geq i \geq I_L \geq 0$), $b > 0$, and for divestment projects ($I_H \leq i \leq I_L \leq 0$), $b < 0$; so in particular, the surplus created by both types of project, $i\theta b$, is positive. For divestments, value creation stems from cash being more valuable in the hands of shareholders than the firm's, either because of internal agency problems in the firm, or because of shareholders' liquidity needs.[7]

A firm's value V is the combination of its assets in place a, the funds raised or disbursed by the equity transaction i, and transaction surplus S:

$$V(a,i,\theta) \equiv a + i + S(i,\theta) = a + i + i\theta b. \tag{1}$$

For divestment projects we further assume that $b > -1$, which ensures that repurchases indeed reduce a firm's total value. Finally, we assume that the minimum firm value is positive after equity transaction, $a_{\min} + I_H > 0$.

The number of shares outstanding before any issue or repurchase is normalized to 1. Given an equity transaction price p, the firm needs to issue $\frac{i}{p}$ shares to raise capital i, or repurchase $\frac{-i}{p}$ shares for $i < 0$ to disburse i. The firm maximizes the payoff of its long-term investors, which is given by

$$\Pi(a,i,\theta,p) = \frac{V(a,i,\theta)}{1+\frac{i}{p}}. \tag{2}$$

Section 5 analyzes the more general case in which firms care about both short- and long-term share prices.

For both intuition and formal analysis, it is frequently convenient to work with the log of the firm's

[7]Concretely, for the repurchase case, consider a firm with cash $|I_H|$ and non-cash assets in place a_N. Cash held inside the firm is invested inefficiently at a random time, which decreases each dollar's value from 1 to $1-|b|$. Absent repurchases, the inefficient investment is made for probability 1. Hence the firm's asset in place is $a \equiv a_N + |I_H|(1-|b|)$. If the firm repurchases $|i|$ with efficieny θ, θ controls the time of the repurchase, and hence the probability that the repurchase occurs before the inefficient investment. Hence the firm's value is

$$\begin{aligned} V(a,i,\theta) &= a_N + \theta\left[\underbrace{(|I_H|-|i|)(1-|b|)}_{\text{remaining cash if } |i| \text{ is paid out before inefficient investment}}\right] \\ &\quad + (1-\theta)\left[\underbrace{|I_H|(1-|b|)-|i|}_{\text{remaining cash if} |i| \text{is paid out after inefficient investment}}\right] \\ &= a_N + |I_H|(1-|b|) - |i| + \theta|i||b| \\ &= a + i + i\theta b, \end{aligned}$$

which coincides with (1).

payoff,

$$\ln \Pi\left(a, i, \theta, p\right) = \ln V\left(a, i, \theta\right) - \ln\left(1 + \frac{i}{p}\right). \tag{3}$$

That is, firms trade off percentage changes in firm value V with percentage changes in the number of shares outstanding after the equity transaction. The fact that it is percentage changes that is important stems from our focus on equity transactions.

By design, this framework covers both issue and repurchase decisions in a symmetric way. For the remainder of the paper, we refer to the case $I_H \geq i \geq I_L \geq 0$, $b > 0$ as the *issue game*, and the case $I_H \leq i \leq I_L \leq 0$, $b < 0$ as the *repurchase game.*

We focus on pure-strategy equilibria, which consist of each firm-type's choices of investment and efficiency, $\left(i\left(a\right), \theta\left(a\right)\right)$; investor beliefs $\mu\left(a|i, \theta\right)$ associated with each choice of $\left(i, \theta\right)$; and competitive investors' pricing function, $P\left(i, \theta\right)$, such that the following three conditions hold:

1. Given $P\left(i, \theta\right)$, firm a's equilibrium strategy $\left(i\left(a\right), \theta\left(a\right)\right)$ maximizes its long-term shareholders' payoff:

$$\left(i\left(a\right), \theta\left(a\right)\right) \in \arg\max_{i,\theta} \Pi\left(a, i, \theta, P\left(i, \theta\right)\right)$$

2. The pricing function $P\left(i, \theta\right)$ is such that investors break even, i.e.,

$$P\left(i, \theta\right) = E\left[\Pi\left(a, i, \theta, P\left(i, \theta\right)\right)|i, \theta\right], \tag{4}$$

where expectations are taken using beliefs $\mu\left(a|i, \theta\right)$. Notice $\Pi\left(a, i, \theta, P\left(i, \theta\right)\right)$ is indeed the true value of each share post transaction.

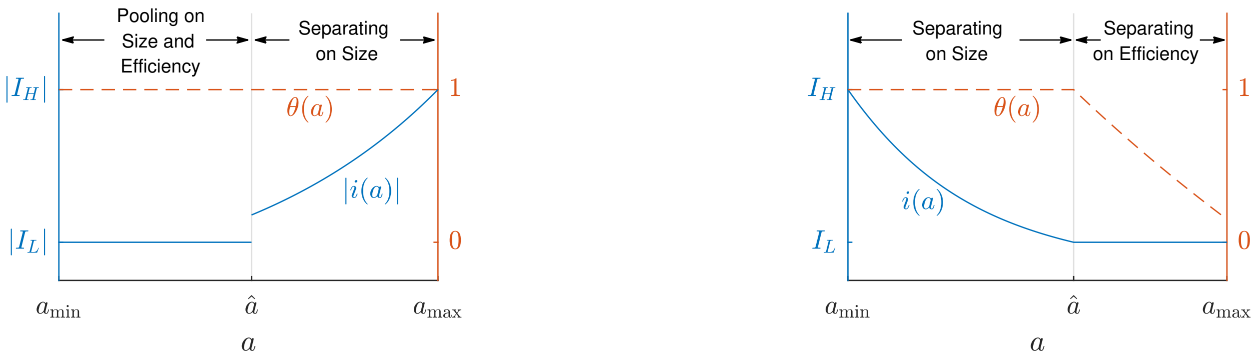
3. Investor beliefs $\mu\left(a|i, \theta\right)$ satisfy Bayes' rule for any $\left(i, \theta\right)$ such that $\left(i, \theta\right) = \left(i\left(a\right), \theta\left(a\right)\right)$ for some firm type a.

As in many signaling models, there are typically multiple equilibria. We employ the widely accepted D1 criterion (Cho and Kreps, 1987) to eliminate equilibria with "unreasonable" off-equilibrium beliefs. Broadly speaking, D1 requires that the beliefs associated with any off-equilibrium action must place all weight on types most likely to deviate to that action. Formally, let $\Pi^*\left(a\right)$ denote the equilibrium payoff of a type a firm. Given an investment and efficiency $\left(i, \theta\right)$, define

$$D_a\left(i, \theta\right) = \left\{p : \Pi\left(a, i, \theta, p\right) > \Pi^*\left(a\right)\right\}. \tag{5}$$

An equilibrium satisfies D1 if for any type a for which there exists a second type $\tilde{a}$ such that $D_a\left(i, \theta\right) \subsetneq D_{\tilde{a}}\left(i, \theta\right)$, beliefs satisfy $\mu\left(a|i, \theta\right) = 0$. It is worth noting that one of our central results—the impossibility of separation-via-efficiency by repurchasing firms—does not rely on equilibrium refinements.

Remark: As noted, we often interpret the efficiency choice θ in terms of how smooth a transaction is. Transactions that are smooth—that is, OMR and ATM programs—also entail optionality, since



(a) The repurchase game. (b) The issue game.

Figure 1: Equilibrium size and efficiency in the repurchase and issue games. Values of $|i(a)|$ and $i(a)$ correspond to the left y-axis, and values of $\theta(a)$ correspond to the right y-axis.

firms can transact smaller quanitites than the initial announcement. However, adding optionality to the model does not change the equilibrium outcomes.[8] For brevity, we abstract from this aspect, and assume that firms issue and repurchase the full amount that is announced.

3 Equilibrium Characterization

We fully characterize the equilibria of the repurchase and issue games. Specifically, for repurchases we show that separation-via-efficiency is impossible; while separation-via-size naturally arises, with worse firms repurchasing less, at lower prices. For issues, firms separate by issuing different quantities, with better firms issuing less, at higher prices; and the best firms further separate by issuing inefficiently, at still higher prices. Figure 1 summarizes these results.

3.1 Full information benchmark

As a benchmark, consider the case in which a firm's assets a are publicly observed. From (1), (2) and (4),

$$P(i,\theta) = \Pi(a,i,\theta,p) = a + S(i,\theta).$$

Hence in this benchmark, and as one would expect, firms choose transaction size i and method θ to maximize transaction surplus $S(i,\theta)$. Firm value $V(a,i,\theta)$ is irrelevant to the decision. Under our specification of S, firms choose $i = I_H$ and $\theta = 1$.

[8]Specifically, we consider the following perturbation of the model. Let $\Theta \subset [0,1]$ be the set of choices of θ that entail optionality. As in the main model, a firm publicly selects (i,θ). Different from the main model, if $\theta \in \Theta$ then the firm can privately deviate to a smaller transaction, viz., choose an actual transaction of $\left|i^A\right| \in [|I_L|, |i|]$. The equilibria in Proposition 3 and 5 are still the unique D1 equilibria of the repurchase and issue games of this perturbed model. See Online Appendix B for details.

3.2 Repurchases

We first analyze the behavior of firms wishing to pay out funds by repurchasing shares. We start by showing that repurchasing firms are unable to separate from each other by repurchasing with different efficiency levels. As we will see, this impossibility of separation-via-efficiency contrasts sharply with the possibility of such separation by issuing firms that seek to raise funds.

Proposition 1. *In the repurchase game, (A) all firms that repurchase the same size i choose the same efficiency θ, and (B) in a D1 equilibrium, all firms that repurchase do so with maximal efficiency $\theta = 1$.*

To understand the economics behind part (A), the impossibility of separation-via-efficiency, start by observing that it is worse firms that wish to reveal their types so as to repurchase shares at lower prices. Focusing on (3), suppose that worse firms attempt to separate by adopting some less efficient method $\tilde{\theta} \equiv \theta - \Delta\theta < \theta$ in exchange for a lower repurchase price $P\left(i, \tilde{\theta}\right)$. On the one hand, the resulting sacrifice in firm value V is $\Delta\theta i b$, which represents a smaller fraction of a better firm. On the other hand, the percentage change in the number of shares is independent of firm type. Consequently, the lower efficiency choice $\tilde{\theta}$ is more attractive for good firms than bad firms, and so separation of this type is impossible in equilibrium.

Part (B) of Proposition 1 likewise follows from the observation that worse firms experience the larger (percentage) effects from changing efficiency. Suppose that, contrary to the claimed result, firms repurchase using an inefficient method $\theta < 1$. Then firms would like to deviate and repurchase more efficiently ($\theta = 1$), provided that doing so doesn't significantly increase the repurchase price. The D1 refinement ensures that this condition is met: deviations to $\theta = 1$ tend to induce price decreases, because it is worse firms who experience the larger (percentage) benefit from repurchasing more efficiently, and so the off-equilibrium-path belief is concentrated on worse firms.

In contrast to the impossibility of separation-via-efficiency, repurchasing firms are able to separate by repurchasing less. We first establish this result for the case in which a firm's informational advantage is limited, in the sense that the support of a is sufficiently small. The economic forces are easiest to describe in this case. Subsequently, we fully characterize the equilibrium incorporating the case where a firm's informational advantage is larger.

Proposition 2. *In a D1 equilibrium of the repurchase game, if $a_{\min}$ and $a_{\max}$ are sufficiently close, then firms separate on transaction size according to strategy $\hat{i}(\cdot)$:*

$$\frac{\partial \hat{i}(a)}{\partial a} = -\frac{\hat{i}(a)}{V\left(a, \hat{i}(a), 1\right) b}, \tag{6}$$

with the boundary condition

$$\hat{i}(a_{max}) = I_H. \tag{7}$$

Why can repurchasing firms separate using size i even though they cannot separate using efficiency θ (Proposition 1)? The reason is that reducing repurchase size increases firm value V while reducing efficiency decreases V—even though transaction surplus is reduced in both cases. In more detail: In the repurchase setting, it is worse firms that wish to separate themselves from better firms so as to be able to acquire shares at a lower price. Consider a firm that offers a smaller repurchase size $|\tilde{i}| \equiv |i| - \Delta i$ for some $\Delta i > 0$, in order to obtain a lower price. While this smaller repurchase lowers transaction surplus by $\theta \Delta i |b|$, it increases firm value by $\Delta i \left(1 + \theta b\right)$, because the firm retains more cash. This increase represents a larger fraction of total value for worse firms. So by (3), a smaller repurchase is more attractive for worse firms, making it a viable signal.

The formal characterization in (6) and (7) of separation via repurchase size is standard (e.g., Mailath (1987)). First, there is no distortion at the "bottom," in this case meaning that the best firm $a_{\max}$ repurchases the maximum amount I_H. Second, worse firms separate by repurchasing less, which has the advantage of reducing the repurchase price. Given separation, repurchases are fairly priced, i.e., $P\left(i\left(a\right), \theta\left(a\right)\right) = a + S\left(i\left(a\right), \theta\left(a\right)\right)$. As standard, the equilibrium condition is that firm a does not gain from mimicking neighboring firms, so that equilibrium strategy $\hat{\imath}\left(a\right)$ solves the differential equation

$$\frac{d}{d\tilde{a}} \Pi \left(a, \hat{\imath}\left(\tilde{a}\right), 1, \tilde{a} + S\left(\hat{\imath}\left(\tilde{a}\right), 1\right)\right)_{\tilde{a}=a} = 0 \tag{8}$$

subject to the boundary condition $\hat{\imath}\left(a_{\max}\right) = I_H$. By straightforward manipulation, (8) simplifies to (6). Note that Proposition 2 builds on Proposition 1's result that firms choose maximal efficiency $\theta = 1$.

Propositions 1 and 2 represent the principle insights of this subsection. First, separation-via-efficiency is impossible for repurchasing firms. It is worse firms that wish to separate to drive down the price, but adopting an inefficient repurchase method is disproportionately costly for such firms. Second, and in contrast, worse firms can separate by scaling down their repurchases. Although scaling down a repurchase reduces the transaction surplus—i.e., "burns money"—just like adopting an inefficient method, doing so *increases* total firm value. The increase in firm value is larger (in percentage terms) for worse firms, rendering it an effective way for such firms to signal.

The remainder of the subsection completes the characterization of repurchase equilibria, specifically, by characterizing repurchase outcomes when firm's informational advantage is larger than the case of Proposition 2.

Equation (6) characterizes the form that separation on repurchase size takes. If $a_{\min}$ is sufficiently close to $a_{\max}$ that (6) leads to repurchases above the minimum size I_L ($|i| > |I_L|$, i.e., $i < I_L$) for all firms $a > a_{\min}$, then Proposition 2 is already a complete description of the repurchase equilibrium. For use in Proposition 3 below, define $\hat{a} = a_{\min}$ in this case.

The remaining case in which $\hat{\imath}\left(\cdot\right)$ hits this minimum repurchase level $I_L < 0$ before $a_{\min}$ is reached is more complicated. As a first step, it is instructive to note that it *cannot* be an equilibrium for separation to continue according to (6) all the way until the minimum repurchase size $I_L < 0$ is hit. The reason is that in such a case, there is an interval of firms below the separating firms that pool on the minimum repurchase size I_L. But firms marginally better than the pooling firms would gain by deviating marginally and reducing their repurchases to I_L, since doing so generates a discrete price reduction.

Instead, the equilibrium consists of a cutoff type $\hat{a}$ that is indifferent between repurchasing $\left|\hat{i}(\hat{a})\right| > |I_L|$ according to (6) and (7) at the separating price $\hat{a} + S\left(\hat{i}(\hat{a}), 1\right)$ and repurchasing $|I_L|$ at the pooling price $a + E\left[S(I_L, 1) | a \in (a_{\min}, \hat{a})\right]$:

$$\hat{a} + S\left(\hat{i}(\hat{a}), 1\right) = \frac{V(\hat{a}, I_L, 1)}{1 + \frac{I_L}{E[a|a\in[a_{\min},\hat{a})] + S(I_L,1)}}. \tag{9}$$

Firms better than $\hat{a}$ separate according to (6). As discussed immediately above, the separation region ends at $\left|\hat{i}(\hat{a})\right|$ before the minimum repurchase I_L is hit. Firms worse than $\hat{a}$ pool and repurchase the minimum amount, I_L; and so in particular, repurchase discretely less than firms better than $\hat{a}$. See Figure 1 for an illustration of this case. If (6) and (7) imply $\hat{i}(a) = I_L$ for some a but there is no $\hat{a}$ that satisfies (9), then the equilibrium is simply that all firms pool and repurchase $|I_L|$, i.e., $\hat{a} = a_{\max}$.

Summarizing:

Proposition 3. *The repurchase game has a unique D1 equilibrium, in which firms with $a > \hat{a}$ separate and repurchase according to $\hat{i}(\cdot)$ defined by (6) and (7), and firms $a < \hat{a}$ pool at the minimum repurchase size I_L. All repurchases take place at maximal efficiency, $\theta = 1$.*

Finally, we characterize the analytical solution to the ODE given by (6) and (7),

$$\left(-\hat{i}(a)\right)^b \left(a + \hat{i}(a)\, b\right) = (-I_H)^b \left(a_{\max} + I_H b\right). \tag{10}$$

In the special case in which the minimum repurchase size is $I_L = 0$, the solution $\hat{i}$ in (10) never reaches $I_L = 0$. In this case, the cutoff type $\hat{a}$ is $a_{\min}$, and all firms separate on their size choice according to $\hat{i}$.

3.3 Issues

We now turn to the behavior of firms wishing to raise funds by issuing shares ($I_H > I_L \geq 0$). We establish a stark asymmetry between the two cases, namely that separation-via-efficiency is feasible for issuing firms even though it is not for repurchasing firms.

To show that separation-via-efficiency occurs for issuing firms, we fully characterize the unique D1 equilibrium of the issue game. We start by showing that although issuing firms separate via both efficiency and size choices, they prefer to do so via size, and use inefficient methods as a signal only when they have exhausted the use of size as a signal.

Proposition 4. *In any D1 equilibrium of the issue game, if a firm issues $i > I_L$ then it uses the most efficient method $\theta = 1$.*

The economic intuition for an issuing firm's preference to separate via size is as follows. Suppose to the contrary that a D1 equilibrium exists in which some firm a issues more than the minimum amount, $i > I_L$, but uses an inefficient method $\theta < 1$. By issuing less but transacting more efficiently, the firm can both increase transaction surplus S and reduce its total value V. That is, there exists a deviation to $\tilde{i} < i$ and $\tilde{\theta} > \theta$ such that

$$\tilde{i}\tilde{\theta}b > i\theta b, \tag{11}$$

$$\tilde{i} + \tilde{i}\tilde{\theta}b < i + i\theta b. \tag{12}$$

The economic principle that makes the combination of (11) and (12) possible is that increasing efficiency (θ) raises transaction surplus and firm value by the same amount; while issuing less leads to a larger reduction in firm value than in surplus.

The percentage reduction in firm value associated with (12) is smaller for better firms. From (3), it follows from D1 that the beliefs associated with this deviation are at least as good as a. So the deviation $\left(\tilde{i}, \tilde{\theta}\right)$ is at least fairly priced for firm a, and since it strictly raises transaction surplus, it strictly raises firm a's payoff.

Given Proposition 4, the structure of the equilibrium of the issue game follows naturally. There is an interval of firms that separate by issue size. This interval is followed by an interval of firms that issue the minimum amount $i = I_L$ and separate by inefficient issue methods.

Taken in isolation, the construction of each of the signaling-via-size and signaling-via-efficiency intervals is standard. The new element in our analysis (relative to, for example, retention signaling models of Leland and Pyle, 1977; Myers and Majluf, 1984; DeMarzo and Duffie, 1999) is to analyze both signaling possibilities together. To reiterate, the key tool is Proposition 4.

The specific form of the issuing equilibrium is as follows. First, there is no distortion at the bottom: the worst firm $a_{\min}$ issues the maximum size ($i = I_H$) at maximum efficiency ($\theta = 1$).

Second, an interval of firms better than $a_{\min}$ separate by scaling down the project, while retaining maximal issue efficiency $\theta = 1$. The construction is the same as for the equilibrium of the repurchase game, with the exception that it starts from the worst firm $a_{\min}$ rather than the best firm $a_{\max}$.

Writing $\hat{i}(a)$ for firm a's issue strategy, the function $\hat{i}(\cdot)$ must solve the differential equation (8), subject to the boundary condition

$$\hat{i}(a_{\min}) = I_H. \tag{13}$$

The economic force behind separation-via-size is similar to in Leland and Pyle (1977), viz., better firms separate by retaining a larger fraction of equity, which is more valuable for them.

Note that although repurchase and issue sizes share the same differential equation (6), the prediction on transaction size is reversed across the two cases, with better firms repurchasing more but issuing less.

Third, separation on issue size according to (6) continues as long as there is room. Specifically, if $\hat{i}(a_{\max}) \geq I_L$, all firms separate on issue size, and the equilibrium characterization is complete; for use in Proposition 5, define $\hat{a} = a_{\max}$. If instead there is a such that $\hat{i}(a) = I_L$, define $\hat{a}$ as the value of a such that $\hat{i}(a) = I_L$.

Firms better than $\hat{a}$ issue the minimum amount I_L, and separate by adopting less efficient methods. Writing $\hat{\theta}(a)$ for firm a's efficiency strategy, for firms $a > \hat{a}$, the equilibrium strategy $\hat{\theta}(a)$ solves the differential equation

$$\frac{d}{d\tilde{a}} \Pi\left(a, I_L, \hat{\theta}(\tilde{a}), \tilde{a} + S\left(I_L, \hat{\theta}(\tilde{a})\right)\right)_{\tilde{a}=a} = 0, \tag{14}$$

subject to the boundary condition $\hat{\theta}(\hat{a}) = 1$. Equation (14) simplifies to

$$\frac{\partial \hat{\theta}(a)}{\partial a} = -\frac{1}{V\left(a, I_L, \hat{\theta}(a)\right) b}. \tag{15}$$

Recall that we assume that the best firm prefers issuing I_L with efficiency $\theta = 1$ under the worst belief to doing nothing (see footnote 5). Under this assumption, there is enough room in efficiency choices θ for all firms better than $\hat{a}$ to fully separate, i.e., $\hat{\theta}(a)$ remains positive for all $a \in [a_{\min}, a_{\max}]$.

Summarizing:

Proposition 5. *The issue game has a unique D1 equilibrium, in which firms with $a \in [a_{\min}, \hat{a}]$ issue $\hat{i}(a)$ in the most efficient way ($\theta = 1$), and firms with $a \in (\hat{a}, a_{\max}]$ issue $i = I_L$ at efficiency $\hat{\theta}(a)$, where $\hat{a}$, $\hat{i}(\cdot)$, and $\hat{\theta}(\cdot)$ are as defined above.*

We highlight that D1 rules out pooling on any issue size or efficiency level. Pooling would necessarily entail some firms issuing at prices below their fair prices. Such firms could profitably deviate to an alternative issue size and efficiency that marginally decreases firm value but discretely improves the issue price. The reason why the issue price improves again follows from (3), i.e., better firms care less in percentage terms about reductions in firm value, and so D1 beliefs about such deviations heavily weight good firms.

Proposition 4 establishes that a necessary condition for firms to separate using transaction efficiency is that the possibilities from separation on size are exhausted. Proposition 5 shows that, for issuing firms, this condition is also sufficient: once the ability to separate via size is exhausted, firms indeed switch to separating via transaction efficiency.

Proposition 4's ordering of signaling-via-size versus signaling-via-efficiency can be understood as operationalizing Viswanathan (1995)'s "benefit-cost criterion." When multiple signaling devices are available, Viswanathan establishes that the Pareto-optimal separating equilibrium uses the signal with the highest "benefit-cost ratio." Formally, define $\pi(a,i,\theta,\tilde{a}) = \ln \Pi(a,i,\theta,\tilde{a}+S(i,\theta))$. Viswanathan's benefit-cost ratios for issue size and efficiency are, respectively, $\frac{-\pi_{ai}(a,i,\theta,a)}{\pi_i(a,i,\theta,a)}$ and $\frac{-\pi_{a\theta}(a,i,\theta,a)}{\pi_\theta(a,i,\theta,a)}$.

At first sight, the comparison of these benefit-cost ratios appears opaque. However, this comparison can be expressed entirely in terms of a signal's effect on firm value and transaction surplus. Specifically:

Lemma 1. *The ordering of ratios* $\frac{-\pi_{ai}(a,i,\theta,a)}{\pi_i(a,i,\theta,a)}$ *and* $\frac{-\pi_{a\theta}(a,i,\theta,a)}{\pi_\theta(a,i,\theta,a)}$ *coincides with the ordering of ratios* $\frac{V_i}{S_i}$ *and* $\frac{V_\theta}{S_\theta}$.

In particular, Lemma 1 formalizes the distinct roles of firm value and transaction surplus in determining a signal's attractiveness.

Precisely because transaction size i affects firm value not only via transaction surplus S but also directly, it is immediate that transaction size has the more attractive benefit-cost ratio,

$$\frac{V_i}{S_i} > \frac{V_\theta}{S_\theta}, \tag{16}$$

consistent with Proposition 4.

Viswanathan (1995) characterizes Pareto-optimal separating equilibria. Abstract papers such as Engers (1987), Cho and Sobel (1990), and Ramey (1996) in turn show that the D1 refinement typically select such equilibria.[9]

Finally, we characterize the analytical solutions to the ODEs (6) (with boundary condition (13)) and (15):

$$\hat{i}(a)^b \left(a + \hat{i}(a)\, b\right) = I_H^b \left(a_{\min} + I_H b\right), \tag{17}$$

[9] For other uses of Pareto-optimality to select among signals in corporate finance settings, see John and Williams (1985), Ambarish, John, and Williams (1987), Besanko and Thakor (1987), Ofer and Thakor (1987), and Williams (1988). Williams (2021) analyzes a seller's choice between signalling via retention and illiquidity in a competitive search model; his results emphasize the role of participation costs of potential investors, which is a dimension that we don't pursue in this paper.

and

$$e^{b\hat{\theta}(a)}\left(a + I_L\hat{\theta}(a)\,b\right) = e^{b}\left(\hat{a} + I_L b\right). \tag{18}$$

Parallel to the repurchase game: When $I_L = 0$, the solution $\hat{\imath}$ in (17) never reaches $I_L = 0$ for any domain $[a_{\min}, a_{\max}]$, and all firms separate on issue size according to $\hat{\imath}$.

4 Empirical Implications

In this section, we explore the empirical implications of our model. Broadly speaking, there are two ways to issue seasoned equity in practice. The first method is a one-off SEO, which is typically completed within several weeks.[10] A lesser known but increasingly popular method is an at-the-market offering (henceforth, ATM). Billett, Floros, and Garfinkel (2019) provide a nice review of ATMs. In an ATM, the firm first registers new shares with the Securities and Exchange Commission (SEC), and then anonymously sells these shares in the secondary market. Compared to SEOs, ATMs take much longer to complete, on average 6.2 quarters. Similarly, firms can repurchase equity in a one-off fashion through a tender offer repurchase (henceforth TOR) within a month.[11] Alternatively, they can carry out an open-market repurchase program (henceforth, OMR) over a horizon of several years.[12]

The starkest prediction to emerge from our analysis (see Propositions 3 and 5) is:

Prediction 1: A greater variety of methods is used in issue transactions than in repurchase transactions.

Consistent with this prediction, significant issuance occurs via both SEOs and ATMs, while an overwhelming fraction of repurchases are OMRs, with only a very small fraction being TORs (see footnote 2).

More specific predictions about transaction methods, and their correlation with transaction size and future outcomes, require us to take a stand of how the efficiency parameter θ in the model maps to different methods. While different assumptions are possible here, we next show that a natural and unified specification based on the availability of cash implies that, for issues, a one-off SEO is more efficient than a smoother ATM, and that for repurchases, a smooth OMR is more efficient than a one-off TOR. In a nutshell: The value creation from issues stems from raising capital to deploy

[10] A non-shelf bookbuilt SEO, which accounts for 91% of all SEOs, often takes 2-8 weeks, while an accelerated bookbuilt SEO often takes 2 days from announcement to completion (Gao and Ritter, 2010; Huang and Zhang, 2011).

[11] It takes 25 days on average from announcement of an TOR to the expiration of the offer (Masulis, 1980).

[12] On average, 46.2%, 66.9%, and 73.9% of the target amount is completed by end of the first, second, and third year, respectively (Stephens and Weisbach, 1998).

in a productive way, without waiting for a firm's retained earnings to accumulate to fund the new investment; and a one-off SEO allows capital to be deployed more quickly than a smoother ATM. Conversely, the value creation from repurchases stems from paying out a firm's earnings to avoid wasteful expenditure inside the firm; and a smoother OMR allows earnings to be paid out as they arrive, as opposed to a one-off TOR that entails retaining earnings inside the firm until a significant quantity has accumulated. Here we implicitly assume that the underlying cash generating process is smooth, and our model hence applies to firms that have regular cash flow to pay out. This feature maps well to large profitable firms, such as large technological companies (e.g., Apple, Amazon, Facebook, and Google), profitable banks (Goldman Sachs, JP Morgan, and Bank of America), among many other companies.[13]

Formally, first consider a firm that encounters an investment opportunity at time 0, but lacks funds to undertake it. The project exhibits constant returns to scale over a range of investment levels. The firm chooses both an investment amount i, i.e., project scale; and a time t to start the project. The project can only start after the firm has raised funds i. If implemented at time 0, the project's NPV for each unit of investment is b. As time passes, the project becomes more and more obsolete (for example, due to the entry of competitors), and the NPV decreases at the rate of α. Defining $\theta(t) = e^{-\alpha t}$, the project NPV is of the form $i\theta(t)b$. As such, an immediate SEO corresponds to the highest efficiency level $\theta = 1$, while smoother ATMs correspond to lower efficiency levels.

Next, consider a firm that generates free cash flows at continuous rate λ over a time interval $[0, T]$. If retained inside the firm, these cash flows are deployed in bad projects, generating a negative rate of return $-\beta < 0$. As such, if no payouts are made over some arbitrary time interval $[0, t]$, then these cash flows accumulate to a date-t value of

$$\int_0^t \lambda e^{-\beta(t-s)} ds = \frac{\lambda}{\beta}\left(1 - e^{-\beta t}\right).$$

Consider the payout policy in which the firm spends $\frac{|i|}{n}$ on repurchases at dates $\frac{T}{N}, 2\frac{T}{n}, \ldots, T$. The firm's date T cash balance is then

$$\sum_{m=1}^{n}\left(\frac{\lambda}{\beta}\left(1 - e^{-\beta\frac{T}{n}}\right) - \frac{|i|}{n}\right) e^{-\beta\left(T - m\frac{T}{n}\right)}. \tag{19}$$

Let a_N denote the firm's illiquid assets, which cannot be diverted to bad projects. By straightforward evaluation of (19), the firm's value under the above payout policy is

$$V = a_N + \frac{\lambda}{\beta}\left(1 - e^{-\beta T}\right) - |i| + |i|\left(1 - \frac{1}{n}\frac{1 - e^{-\beta T}}{1 - e^{-\beta\frac{T}{n}}}\right). \tag{20}$$

[13] Another motive to carry out share repurchases is to distribute one-off windfall cash, such as lump-sum damage awards from legal disputes, spinoff proceeds, sudden capital structure adjustment, and so on. But these are rare events, and we believe our model captures the majority of repurchase activities.

Defining $a = a_N + \frac{\lambda}{\beta}\left(1 - e^{-\beta T}\right)$, $b = -\left[1 - \frac{1}{\beta T}\left(1 - e^{-\beta T}\right)\right]$ and $\theta\left(n\right) = \frac{1 - \frac{1}{n}\frac{1-e^{-\beta T}}{1-e^{-\beta\frac{T}{n}}}}{|b|}$, (20) coincides with our general specification of firm value (1). Note that θ increases monotonically from 0 to 1 as n increases from 1 (single repurchase at date T) to ∞ (continuous repurchases over $[0, T]$). As such, smoother repurchases correspond to higher efficiency levels θ.

This formalization allows us to replace Prediction 1 with the more specific:

Prediction 1': Firms issue equity using both SEOs and ATMs, while firms repurchase equity via OMRs.

Relative to Prediction 1, the incremental insight of Prediction 1' is that firms repurchase using OMRs. Empirically, this is overwhelmingly the case.

Proposition 5 also delivers the cross-sectional prediction that a firm carries out larger issues using efficient methods, which in our interpretation corresponds to an SEO. In contrast, for smaller issues a firm sometimes uses more inefficient methods, corresponding to ATM issues:

Prediction 2: SEOs are larger than ATM programs.

Empirically, Billett, Floros, and Garfinkel (2019) document average SEO proceeds of \$256 million, compared to average ATM program proceeds of \$92 million.[14]

Proposition 5 also implies:

Prediction 3: Firms with better unobservable qualities are more likely to use ATM issues.

Prediction 4: Firms facing larger informational frictions are more likely to use ATM issues.

In Prediction 4, the size of the informational friction is captured by the dispersion of firm types, $a_{\max} - a_{\min}$. Proposition 5 predicts that separation via transaction efficiency arises only when the dispersion of firm types is large.

Consistent with Prediction 3, Hartzell et al. (2019) shows in a dataset of REITs that the announcement returns of ATMs are less negative than of SEOs.[15]

Billett, Floros, and Garfinkel (2019) use *future* analyst recommendation updates as their proxy for firm quality unobservable to the market at the time of issuance. Consistent with our Prediction 3, their regression result in Table 4 shows that ATM firms receive better future analyst recommendation

[14]See Table 2 in Billett, Floros, and Garfinkel (2019). Table 4 in the same paper reports regression results after controlling for additional observable factors, including size of the issuing firm, and likewise indicates that SEOs are larger.

[15]That both ATMs and SEOs are followed by negative returns can be generated in our model by relaxing the assumptions of footnote 5, to allow for the possibility that the best firms do not issue any shares.

updates than SEO firms. Consistent with Prediction 4, the same authors show that higher levels of information asymmetry, proxied by unexplained current accruals, are indeed associated with the choice of ATM over SEO.

We close this section with a brief discussion of an implication that superficially appears more testable than we believe it is, viz., the prediction of pooling at the minimum transaction size I_L. The difficulty of testing this prediction is that I_L—in common with all parameters of the model—is common knowledge, and so should be understood as being a function of observable firm characteristics. As such, the prediction of pooling at I_L could only be tested by an econometrician who knows how I_L varies with firm characteristics.

5 Preference for Share Price

Many commentators suggest that when public firms repurchase shares they are motivated primarily by a desire to boost their share prices.[16]

In contrast, our analysis so far has followed the standard modelling assumption that firms aim to maximize the interest of long-term shareholders[17]. As discussed, this assumption implies that, ceteris paribus, firms prefer lower rather than higher share prices when repurchasing shares. In this section, we analyze an extension of our baseline model in which firms care about *both* short-term share prices and the interests of long-term shareholders. This analysis suggests that the weight that firms put on short-term share prices is at most modest. Specifically, we show that an assumption that firms heavily weight short-term share prices generates counterfactual predictions, discussed below.

Formally, we generalize (2) by assuming that firms maximize a geometrically weighted average of the short-term share (transaction) price p and the long-term shareholders' payoff:

$$\Pi(a, i, \theta, p) = p^{\epsilon} \left(\frac{V(a, i, \theta)}{1 + \frac{i}{p}} \right)^{1-\epsilon}, \tag{21}$$

where the weight $\epsilon \in [0, 1]$ reflects the degree to which firms care about their share prices directly. When $\epsilon = 0$, this objective function reduces to the baseline (2). All other ingredients are the same as in the baseline model.

We first consider the repurchase game. The interesting case is when ϵ is sufficiently large. In this case, the firm may favor a high repurchase price, even though buying more expensive shares hurts

[16]For instance, Segal (2023) writes in an article on investopedia.com that one of the reasons that a company buy back its own shares is "boosting its financial ratios". See also Dittmar (2000) for a review of motivations for share repurchases. Many politicians have denounced share repurchases by invoking similar ideas.

[17]See, for example, Myers and Majluf (1984), Constantinides and Grundy (1989), and Chowdhry and Nanda (1994).

the long-term shareholders. As a result of this shift in preference over the transaction price, the equilibrium outcome qualitatively changes compared with the baseline model. All firms pool on the same repurchase size and separate by choosing different methods. In particular, good firms signal their values by using inefficient methods.

Proposition 6. *If $\epsilon > \frac{-I_H}{a_{\min}}$, then the repurchase game has a unique D1 equilibrium, in which all types repurchase the maximum amount I_H. There is a cutoff type $\hat{a}$ such that firms with $a < \hat{a}$ separate on methods according to $\check{\theta}(\cdot)$ defined by*

$$\frac{d\check{\theta}(a)}{da} = -\frac{\epsilon V(a, I_H, \theta) + (1-\epsilon)I_H}{V(a, I_H, \theta)I_H b} \tag{22}$$

and boundary condition $\check{\theta}(a_{\min}) = 1$. Firms with $a > \hat{a}$ use the least efficient method $\theta = 0$.

In contrast to the baseline model ($\epsilon = 0$) where separation on efficiency θ is impossible, when firm's overall preference is for a higher repurchase price, separation on size becomes impossible, and the only feasible signal is efficiency. Similar to before, this result does not rely on D1 refinement. Intuitively, this contrast is because the single-crossing property remains the same as in the baseline model, whereas the preference over the repurchase price is reversed. Good firms prefer to reveal their superior quality and receive a higher price by sacrificing NPV – either a smaller repurchase or less efficient method. Suppose good firms attempt to separate on size by repurchasing less and retaining more cash. Since the retained cash is a larger fraction of bad firms, they are more willing to adopt a smaller size and hence will mimic good firms. In contrast, separation on transaction efficiency is feasible, since a sacrifice in efficiency reduces firm value and is relatively more costly for bad firms.

On the other hand, when ϵ is small in the repurchase game, or in the issue game with any $\epsilon \in [0, 1]$, equilibrium outcomes are qualitatively similar to that in the baseline model.[18] This is hardly surprising because firms' preference in these situations is similar to that in the baseline model. When ϵ is small in the repurchase game, firms put little weight on direct preference for a high repurchase price, and their incentive to repurchase shares cheaply dominates. In the issue game, firms prefer to issue at a high price even without an explicit preference for price ($\epsilon = 0$). Therefore, introducing a positive ϵ only strengthens this preference, and the equilibrium outcome does not change qualitatively.

[18] Analogous to footnote 5, we make the following simplifying assumptions to ensure all types participate: In the issue game,

$$\Pi\left(a_{\max}, I_L, 1, a_{\min} + S\left(I_L, 1\right)\right) > a_{\max}.$$

In the repurchase game,

$$\Pi\left(a_{\min}, I_H, 1, a_{\max} + S\left(I_H, 1\right)\right) > \Pi\left(a_{\min}, 0, 1, a_{\max}\right).$$

Proposition 7. *With $\epsilon < \frac{-I_H}{a_{\max}+I_H b}$, D1 equilibria of the repurchase game exist. In any D1 equilibrium, firms' strategy has the same format as in the D1 equilibrium with $\epsilon = 0$ except that ODE (6) generalizes to*

$$\frac{d\hat{i}(a)}{da} = -\frac{\epsilon V(a, \hat{i}, 1) + (1-\epsilon)\hat{i}}{V(a, \hat{i}, 1)b}. \tag{23}$$

If $\epsilon \leq \frac{-I_L}{E[a]+I_L b}$ is also satisfied, the value of $\hat{a}$ and hence the D1 equilibrium is unique.

With $\epsilon \in [0, 1]$, the issue game has a unique D1 equilibrium, in which firms' strategy is the same as in the D1 equilibrium with $\epsilon = 0$ except that ODE (6) generalizes to (23) and ODE (15) generalizes to

$$\frac{d\hat{\theta}(a)}{da} = -\frac{\epsilon V(a, I_L, \hat{\theta}) + (1-\epsilon)I_L}{V(a, I_L, \hat{\theta})I_L b}. \tag{24}$$

The sharp contrast between the big- and small-ϵ cases in repurchases sheds light on firms' objective when conducting repurchases. On the one hand, should firms conduct repurchases to boost current share price as many financial commentators and politicians have argued, then ϵ is large (perhaps even close to 1), and Proposition 6 implies firms tend to repurchase similar amounts but adopt various methods with different efficiencies. On the other hand, if firms care predominantly about their long-term shareholders' payoff, Proposition 7 implies the opposite: firms separate on size but not on methods with different efficiency levels. Empirically, repurchase size as a fraction of the firm's market value is rather dispersed: Ikenberry, Lakonishok, and Vermaelen (1995) document respectively 20%, 30%, 31% and 19% of repurchases are 0-2.5%, 2.5-5%, 5-10% and above 10% of the firm value. Moreover, firms tend to pool on the repurchase method as 95% of the repurchases are carried out in the form of an open market repurchase program. These observations suggest that repurchasing firms' main objective is likely to maximize long-term shareholders' value rather than boosting share prices in the short run.[19]

When firms issue equity, the ϵ-preference on share price only has a quantitative impact on the equilibrium outcome. The direct preference on share price (a bigger ϵ) strengthens firms' preference for a higher issue price, resulting in stronger signaling incentives. As a result, the equilibrium features higher costs of signaling, reflected by firms' choices of lower issue size and efficiency. Let $i(a; \epsilon)$ and $\theta(a; \epsilon)$ denote firms' equilibrium choice of issue size and efficiency given the preference parameter ϵ.

Corollary 1. *Firms' equilibrium issue size strategy $i(a; \epsilon)$ and efficiency strategy $\theta(a; \epsilon)$ are non-increasing in ϵ.*

We conclude our discussion of Propositions 6 and 7 with a technical note on the parameter range of ϵ. We do not fully characterize the equilibrium outcomes of the repurchase game for $\epsilon \in$

[19] Dittmann et al. (2022) reach the same conclusion using a completely independent approach.

$\left[\frac{-I_H}{a_{\max}-I_H b}, \frac{-I_H}{a_{\min}}\right]$. The difficulty is that for these intermediate values of ϵ, the firms' overall preference on share price (the sign of $\frac{\partial \Pi(a,i,\theta,p)}{\partial p}$) is indeterminate and varies with the transaction size and efficiency. In addition, when $\epsilon \in \left(\frac{-I_L}{E[a]-I_L b}, \frac{-I_H}{a_{\max}-I_H b}\right)$, there might be multiple equilibria due to the fact that the boundary indifference condition that pins down $\hat{a}$ may have multiple solutions.[20]

6 Private Information on Project Profitability

In our main analysis, a firm's private information is about its assets in place a. Here, we summarize the outcomes that arise under the alternative assumption that a firm instead has private information about the profitability of the project to be implemented, i.e., b. A firm's post-transaction value continues to be given by (1). For comparability with our main model, we assume that it is common knowledge that the project is profitable, i.e., $b > 0$ for investment projects and $b < 0$ for divestment projects, and that private information is solely about the level of b.

The implications of this alternative assumption are easy to summarize (see Proposition 8): No signaling occurs, and all firms undertake the project at its maximal scale, with maximal efficiency $\theta = 1$. This result in turn suggests that, in practice, firms have significant private information about their assets in place a; if instead all their private information concerns project profitability b then it is hard to account for the empirical observations that market reactions to issues and repurchases grow in transaction size.[21]

We sketch the argument here, while relegating all details to the online appendix. For the issue game, this result is close to analysis in the existing literature. In brief: It is firms with better projects (higher b) that wish to signal their type in order to increase the issue price. But both reducing project size and adopting inefficient methods are more expensive for firms with more profitable projects, and so neither approach is a viable signal.

We now turn to the repurchase game. As a starting point, recall from footnote 7 that a_N denotes

[20] The cutoff type $\hat{a}$ satisfies one of the following:

1. $\hat{a} = a_{\min}$ *and* $f(a_{\min}) \leq 0$,
2. $\hat{a} \in (a_{\min}, a_{\max})$ *and* $f(\hat{a}) = 0$,
3. $\hat{a} = a_{\max}$ *and* $f(a_{\max}) \geq 0$

for

$$f(\tilde{a}) \equiv \Pi\left(\tilde{a}, I_L, 1, E\left[a | a \in (a_{\min}, \tilde{a})\right] + S(I_L, 1)\right) - \tilde{a} - S\left(\hat{\imath}(\tilde{a}), 1\right).$$

When $\epsilon > \frac{-I_L}{E[a]-I_L b}$, the three cases may not be mutually exclusive, and the second case may be satisfied by multiple values of $\hat{a}$.

[21] This finding significantly extends Myers and Majluf's (1984) observation that, in their model with fixed issuance size and efficiency, if a firm's private information is only about project profitability b then all firms issue and no signalling occurs.

a firm's non-cash assets-in-place, and post-transaction firm value is

$$V = a_N + (|I_H| - |i|) - |b| (|I_H| - \theta |i|) . \tag{25}$$

In words: the value created by repurchases stems from mitigating wasteful spending inside the firm, and so firms with more to gain from repurchases (higher $|b|$) are less valuable (for any given size and efficiency of repurchase).

Because of this natural property, firms that benefit most from repurchases (high $|b|$) would like to signal their types in order to reduce repurchase prices. But a reduction in the efficiency of repurchases leads to an especially large percentage reduction in the post-transaction value of such firms, and so cannot serve as a signal.

Can a smaller repurchase be used as a signal? After all, in the case in which a firm's private information is about assets in place, repurchase size is a viable signal even though efficiency is not, because of the fact that smaller repurchases increase firm value by a larger percentage for low-valued firms. However, if private information is about the benefit $|b|$ of repurchases, this same force acts against the signaling role of smaller repurchase sizes, because in this case a smaller repurchase disproportionately boosts the firm value of a low-$|b|$ firm. To see this, note that a small reduction in repurchase size affects firm value by[22]

$$-\frac{\partial \ln V}{\partial |i|} = \frac{1 - |b|}{a_N + (1 - |b|) (|I_H| - |i|)}. \tag{26}$$

The denominator embodies the same effect that arises when private information is about assets in place: one dollar is a larger proportion of the firm value of a worse firm (here, higher $|b|$). In contrast, the numerator embodies an offsetting effect specific to private information about project value b: reducing repurchases destroys more project surplus for worse firms (higher $|b|$), and hence the increase in firm value due to retained cash is smaller for these firms. The latter effect is the dominant one, so that (26) is decreasing in $|b|$, i.e., it is better firms (lower $|b|$) that experience the greatest proportional increase in firm value from repurchasing less. Consequently, a worse firm (higher $|b|$) is unable to signal its type by reducing repurchases.

Formally, we establish:[23]

[22] Here, we are using $\theta = 1$, which is the efficiency level chosen by all firms as implied by the D1 refinement.

[23] Proposition 8 is established under the assumption that transaction surplus S is multiplicatively separable in a firm's observable efficiency choice and its privately observed project profitability b. This property arises naturally for the issue game, and holds in the repurchase game for the microfoundation of footnote 7. However, this property isn't satisfied by the microfoundation of Section 4. For this latter case, we can still establish a version of Proposition 8 provided that the support of $|b|$ is a sufficiently small neighborhood $\left[0, \bar{b}\right]$. Moreover, since the proof boils down to showing $\frac{\partial^2 S(\beta,i,n)}{\partial n \partial \beta} > 0$ for all n and β, which is independent of the parameters I_H, I_L and a_N, we have numerically verifed that the result extends to the full support of n and the support of β being as wide as $[0, 0.49]$. See Online Appendix D for details.

Proposition 8. *If the firm privately knows b, while a is public information in the issue game, and a_N is public information in the repurchase game, then in both the issue game and the repurchase game, the unique D1 equilibrium is that all firms undertake the maximum transaction ($i = I_H$) at maximal efficiency.*

7 Discussion

7.1 Transaction fees

One of our key theoretical insights is that when firms issue or repurchase equity, signaling through cash burning (inefficient transaction methods) is fundamentally different from signaling through reduction in transaction size. Some other actions can be modeled as combinations of these two signals, and hence can be incorporated and understood in our framework. For example, a firm paying a transaction fee c out of the amount of cash to pay out $|i|$ can be interpreted as simultaneously burning repurchase surplus by c and reducing repurchase size to $|i| - c$.[24] If firms can choose the amount of transaction fee c (with i and θ fixed), they may separate on different transaction fees in equilibrium due to the effect of c on repurchase sizes. But when firms can directly choose different repurchase sizes, they separate only on repurchase sizes and pool on these other dimensions such as transaction fees.

7.2 Policy implications

ATM offerings were rarely used until two regulatory changes in 2005 and 2008 made them more accessible.[25] Since then, the use of ATMs has risen sharply (Billett, Floros, and Garfinkel, 2019). The regulatory changes reflected the SEC's intention to "allow more companies to benefit from

[24] Long-term shareholder value in this case will be

$$\begin{aligned}\Pi(a,i,\theta,c,p) &= \frac{V(a,i,\theta)}{1-\frac{|i|-c}{p}} \\ &= \frac{a-|i|+|i|\,\theta\,|b|}{1-\frac{|i|-c}{p}} \\ &= \frac{a-(|i|-c)+(|i|\,\theta\,|b|-c)}{1-\frac{|i|-c}{p}}.\end{aligned}$$

To understand the first equation, notice that the after-repurchase firm value is not affected by c and is still $V(a,i,\theta)$, because the microfoundation in footnote 7 still applies.

[25] In 2005, the SEC Securities Offering Reform (SOR) liberalized the filing requirements when firms "take securities down" from a "shelf registration" of equity offerings, which allowed takedowns to be done without review or delay by the SEC. This opened the door to ATMs, which involve frequent takedowns off a shelf. In 2008, the SEC expanded the eligibility of shelf offerings including ATMs to firms with public floats under $75 million.

the greater flexibility and efficiency in accessing the public securities market..."[26] In line with this intention, Gustafson and Iliev (2017) find that after the 2008 deregulation, the treated firms (listed firms with public floats under \$75 million) raised more public equity and increased their capital expenditure.

Our model implies that even though lifting the barriers to ATMs may allow firms to invest more by issuing more equity, the total surplus (welfare) may decrease.[27] To show this, here we compare the result of our original issue model in Section 3 where both SEOs and ATMs are available issue methods to the outcome under the alternative assumption that ATMs are banned.

As explained in Section 4, we interpret ATMs as issue methods that are less efficient than SEOs, because in an ATM shares are dribbled out, delaying firm's investment. If ATMs are banned in the issue game, firms have SEOs as the only available issue method, which corresponds to $\theta = 1$. In this case, there is a cutoff firm type $\check{a} \leq \hat{a}$ ($\hat{a}$ is defined in Section 3) such that firms with $a \in [a_{\min}, \check{a}]$ take the same actions as in the original model, separating on issue sizes. Different from the original model, firms with $a \in (\check{a}, a_{\max}]$ pool on the minimum issue size I_L.

Proposition 9. *The issue game with θ restricted to 1 has a unique D1 equilibrium. There is a firm type $\check{a} \leq \hat{a}$ such that firms with $a \in [a_{\min}, \check{a}]$ issue $\hat{i}(a)$ as defined by (6) and (13), and firms with $a \in (\check{a}, a_{\max}]$ issue $i = I_L$. $\check{a} < \hat{a}$ if and only if $\hat{a} < a_{\max}$.*

The payoffs of firms with $a \in (\check{a}, \hat{a}]$ are higher than in the issue game where firms can choose θ from $[0, 1]$.

Compared with this case, in the original model where both SEOs and ATMs are allowed, all firms choose weakly larger issue and investment sizes, and weakly less efficient issue methods. In particular, firms with $a \in (\check{a}, \hat{a})$ issue and invest $\hat{i}(a) > I_L$ instead of I_L, which is consistent with the aforementioned empirical findings of Gustafson and Iliev (2017). On the other hand, when ATMs are available, firms with $a \in (\hat{a}, a_{\max}]$ use inefficient ATMs to signal their types, whereas when ATMs are banned, they pool on full efficiency.[28] #Consequently, the net effect of allowing ATMs on total surplus is unclear. However, all the numerical examples we have solved suggest that allowing ATMs shifts the equilibrium to one in which the total surplus is lower.[29]

[26] See the final rule titled "Revisions to the Eligibility Requirements for Primary Security Offerings on Form S-3 and F-3," SEC File No. S7-10-07, December 27, 2007.

[27] Beyond making ATMs practically feasible, the 2005 and 2008 SEC rules also relaxed restrictions on shelf offerings in general. Shelf offerings include shelf SEOs and ATM offerings. Our discussion here is around the effects of allowing ATMs alone.

[28] We have been focusing on model parameters such that all firms prefer issuing shares over doing nothing even under the worst market belief (see footnote 5). For other parameters, allowing ATMs could have the additional effect of allowing firms who would otherwise avoid issuing underpriced shares to separate via ATMs and issue at the fair price.

[29] In the numerical examples we have solved, $a_{\min} = 2$, $a_{\max} = 3$, $I_L = 0.8$, $I_H = 1.2$, and $b = 0.5$. We have varied the distribution of $a - a_{\min}$ among Uniform $(0, 1)$, Beta $\left(\frac{3}{2}, 1\right)$, Beta $(2, 1)$, Beta $(3, 1)$, Beta $(1, 2)$, Beta $\left(1, \frac{5}{2}\right)$, Beta $(1, 3)$, Beta $(1, 4)$, Beta $(1, 5)$ and Beta $(1, 6)$.

Considering the policy implication on each type of firms' payoff, allowing ATMs decreases the payoffs of intermediate firms with $a \in (\check{a}, \hat{a}]$. These firms prefer the outcome of the no-ATM economy because there they pool with better firms instead of separating and issuing at the fair price. In all the numerical examples we have solved, the best type $a_{\max}$ and an interval of types around it have higher payoff when ATMs are allowed, in which case they separate on ATMs rather than pooling with lower types.

8 Conclusion

We analyze issue and repurchase transactions side-by-side, under the assumption that firms have superior knowledge about their values, and can choose both transaction size and method. The comparison of issues and repurchases is new to the literature, and yields fresh insights. First: Despite the conceptual symmetry between issue and repurchase transactions, their equilibrium outcomes aren't mirror images of each other. In particular, repurchasing firms cannot signal via the efficiency of transaction method while issuing firms can. Second, and in contrast, reducing repurchase sizes is a viable signal for repurchasing firms, just as reducing issue size is a viable signal for issuing firms. These implications fit well with empirical evidence. Third, and more conceptually, our analysis isolates a precise formal role for total firm value in equity transactions. Finally, a combination of observed empirical regularities with the predictions emerging from extensions of our baseline model suggests that firms' repurchase decisions are driven primarily by a desire to create value for long-term shareholders, and that a significant fraction of firms' private information is about their assets in place.

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Appendix

A Proofs of Section 3

Lemma 2. *In both issue and repurchase games, if $\tilde{a} > a$ and $V(a, i_1, \theta_1) < V(a, i_2, \theta_2)$, then $\Pi(a, i_1, \theta_1, p_1) \geq \Pi(a, i_2, \theta_2, p_2)$ implies $\Pi(\tilde{a}, i_1, \theta_1, p_1) > \Pi(\tilde{a}, i_2, \theta_2, p_2)$.*

If instead $V(a, i_1, \theta_1) = V(a, i_2, \theta_2)$, then $\Pi(a, i_1, \theta_1, p_1) \geq \Pi(a, i_2, \theta_2, p_2)$ if and only if $\Pi(\tilde{a}, i_1, \theta_1, p_1) \geq \Pi(\tilde{a}, i_2, \theta_2, p_2)$.

Proof. (3) implies

$$\ln \Pi(a, i_1, \theta_1, p_1) - \ln \Pi(a, i_2, \theta_2, p_2) = \ln \frac{V(a, i_1, \theta_1)}{V(a, i_2, \theta_2)} - \ln \frac{1 + \frac{i_1}{p_1}}{1 + \frac{i_2}{p_2}} \geq 0. \tag{A.1}$$

If $\frac{V(a,i_1,\theta_1)}{V(a,i_2,\theta_2)} \in (0, 1)$, then this ratio increases in a. Therefore, for $\tilde{a} > a$, $\ln \Pi(a, i_1, \theta_1, p_1) - \ln \Pi(a, i_2, \theta_2, p_2) > 0$.

If $V(a, i_1, \theta_1) = V(a, i_2, \theta_2)$, then $V(\tilde{a}, i_1, \theta_1) = V(\tilde{a}, i_2, \theta_2)$ holds for any $\tilde{a}$ due to linearity, and

$$\begin{aligned} \ln \Pi(a, i_1, \theta_1, p_1) - \ln \Pi(a, i_2, \theta_2, p_2) &= -\ln \frac{1 + \frac{i_1}{p_1}}{1 + \frac{i_2}{p_2}} \\ &= \ln \Pi(\tilde{a}, i_1, \theta_1, p_1) - \ln \Pi(\tilde{a}, i_2, \theta_2, p_2) \end{aligned}$$

completing the proof. □

Lemma 3. *In an equilibrium, the price associated with (i, θ) satisfies D1 if and only if (1)*

$$P(i, \theta) \geq a + S(i, \theta) \tag{A.2}$$

for any firm type a whose equilibrium choice $(i(a), \theta(a))$ is such that

$$V(a, i, \theta) < V(a, i(a), \theta(a)), \tag{A.3}$$

and (2)

$$P(i, \theta) \leq a + S(i, \theta) \tag{A.4}$$

for any firm type a whose equilibrium choice $(i(a), \theta(a))$ *is such that*

$$V(a, i, \theta) > V(a, i(a), \theta(a)). \tag{A.5}$$

Proof. We first show the "only if" part.

In an equilibrium, fix (i, θ) and a whose choice $(i(a), \theta(a))$ satisfies (A.3). Our goal is to show that any type $\tilde{a} < a$ cannot be associated with (i, θ) under D1, which would then (A.2).

Fix $\tilde{a} < a$. Let p be a price such that if (i, θ) is priced p, type $\tilde{a}$ weakly prefers to deviate:

$$\Pi(\tilde{a}, i, \theta, p) \geq \Pi^*(\tilde{a}).$$

In equilibrium, type $\tilde{a}$ cannot benefit from mimicking type a, that is

$$\Pi^*(\tilde{a}) \geq \Pi(\tilde{a}, i(a), \theta(a), P(i(a), \theta(a))),$$

which implies

$$\Pi(\tilde{a}, i, \theta, p) \geq \Pi(\tilde{a}, i(a), \theta(a), P(i(a), \theta(a))).$$

Lemma 2 implies type a strictly prefers to deviate to (i, θ) at price p, that is

$$\Pi(a, i, \theta, p) > \Pi(a, i(a), \theta(a), P(i(a), \theta(a))) = \Pi^*(a).$$

This implies $D_{\tilde{a}}(i, \theta) \subsetneq D_a(i, \theta)$, that is a is more likely to deviate to (i, θ). D1 therefore implies that (i, θ) cannot be associated with type $\tilde{a} < a$.

The other case (A.5) is similar. The same logic yields that a is more likely to deviate to (i, θ) than $\tilde{a} > a$, that is $D_{\tilde{a}}(i, \theta) \subsetneq D_a(i, \theta)$. D1 implies (i, θ) cannot be associated with $\tilde{a} > a$, and hence (A.4).

Next, we prove the "if" part.

Fix (i, θ), and fix an equilibrium in which (A.2) holds for all a such that (A.3) and (A.4) holds for all a such that (A.5). This implies for the a^* such that

$$P(i, \theta) = a^* + S(i, \theta),$$

(A.3) holds for all $a < a^*$ and (A.5) holds for all $a > a^*$. To prove this price satisfies D1, it is sufficient to show that for any $\tilde{a} \neq a^*$ and any p,

$$\Pi(\tilde{a}, i, \theta, p) \geq \Pi^*(a)$$

implies

$$\Pi\left(a^{*}, i, \theta, p\right) \geq \Pi^{*}\left(a^{*}\right). \tag{A.6}$$

We have proved earlier that for any $\tilde{a} < a < a^{*}$, (A.5) implies

$$\Pi\left(a, i, \theta, p\right) \geq \Pi^{*}\left(a\right). \tag{A.7}$$

Notice $\Pi\left(a, i, \theta, p\right)$ is continuous in a, and hence $\Pi^{*}\left(a\right) = \max_{i,\theta} \Pi\left(a, i, \theta, P\left(i, \theta\right)\right)$ is continuous in a. Therefore, that (A.7) holds for all $a \in \left(\tilde{a}, a^{*}\right)$ implies (A.6).

Similarly, we have proved for any $\tilde{a} > a > a^{*}$, (A.3) implies (A.7), and hence implies (A.6). This completes the proof. $\square$

Proof of Proposition 1:

We first show that all types that repurchase the same size choose the same efficiency. Suppose otherwise that (i, θ) and $(i, \tilde{\theta})$ are equilibrium strategies adopted by two nonempty sets of firms A and $\tilde{A}$, respectively. The transaction prices are therefore

$$p = E\left[a | a \in A\right] + S\left(i, \theta\right)$$

and

$$\tilde{p} = E\left[a | a \in \tilde{A}\right] + S\left(i, \tilde{\theta}\right).$$

Without loss of generality, we assume $\tilde{\theta} < \theta$. Since $\tilde{A}$ firms prefer $(i, \tilde{\theta})$ over (i, θ), Lemma 2 implies that all $a > \inf \tilde{A}$ must share the same preference strictly. Hence, $\sup A \leq \inf \tilde{A}$, and consequently $E\left[a | a \in A\right] < E\left[a | a \in \tilde{A}\right]$. However, this implies type $E\left[a | a \in \tilde{A}\right]$ strictly prefers (i, θ) to $\left(i, \tilde{\theta}\right)$:

$$\begin{aligned}
\Pi\left(E\left[a | a \in \tilde{A}\right], i, \theta, p\right) &> \Pi\left(E\left[a | a \in \tilde{A}\right], i, \theta, E\left[a | a \in \tilde{A}\right] + S\left(i, \theta\right)\right) \\
&= E\left[a | a \in \tilde{A}\right] + S\left(i, \theta\right) \\
&> E\left[a | a \in \tilde{A}\right] + S\left(i, \tilde{\theta}\right) \\
&= \Pi\left(E\left[a | a \in \tilde{A}\right], i, \tilde{\theta}, \tilde{p}\right).
\end{aligned}$$

where the first inequality uses the fact that $\Pi\left(a, i, \theta, p\right)$ is decreasing in p in the repurchase game. Lemma 2 further implies that there is a type in $\tilde{A}$ that strictly prefers (i, θ) to $\left(i, \tilde{\theta}\right)$. Contradiction! Hence, there is no separation-via-efficiency given repurchase size.

We next show that in a D1 equilibrium, all repurchasing firms choose $\theta = 1$. Suppose in contrast, (i, θ) with $\theta < 1$ is an equilibrium strategy associated with the set of firms A. Consider any firm type $a \in A$ such that $a \leq E(A)$ and the potential deviation to a more efficient strategy $(i, 1)$. Lemma 3 implies

$$P(i,1) \leq a + S\,(i,1)\,.$$

However, under such belief, type-a firm strictly prefers $(i, 1)$ to (i, θ):

$$\begin{aligned}
\Pi\,(a,i,1,P(i,1)) &> \Pi\,(a,i,1,a+S\,(i,1)) \\
&= a + S\,(i,1) \\
&> a + S\,(i,\theta) \\
&= \Pi\,(a,i,\theta,a+S\,(i,\theta)) \\
&> \Pi\,(a,i,\theta,P\,(i,\theta))\,.
\end{aligned}$$

This contradiction completes the proof.

Lemma 4. *In an equilibrium of the issue or repurchase game, if there is an interval of firm types A on which the size and method choices $i(a)$ and $\theta(a)$ are continuous, and the choices fully reveal the firm types, then*

$$\frac{d\,(i(a)\theta(a)b)}{da} = -\frac{i(a)}{V\,(a,i(a),\theta(a))}. \tag{A.8}$$

Conversely, if on an interval of firm types A, (A.8) holds, $\frac{di(a)}{da} \leq 0$ in the issue game, $\frac{d\theta(a)}{da} = 0$ in the repurchase game, and price is fully revealing, then no type in A has an incentive to mimic another type in A.

Proof. In an equilibrium, let A be an interval of firm types on which $i(a)$ and $\theta(a)$ are continuous and price is fully revealing. This implies that for all $a \in A$,

$$\Pi^*(a) = a + S\,(i(a),\theta(a))\,.$$

Consider any two firm types $a_1, a_2 \in A$ such that $[a_1, a_2] \subset A$. Equilibrium conditions imply

$$\Pi\,(a_1, i(a_2), \theta(a_2), P^*\,(i(a_2),\theta(a_2))) \leq \Pi^*(a_1), \tag{A.9}$$

$$\Pi\,(a_2, i(a_1), \theta(a_1), P^*\,(i(a_1),\theta(a_1))) \leq \Pi^*(a_2). \tag{A.10}$$

Using the functional form of Π and V, condition (A.9) can be explicitly written as

$$\begin{aligned}
& (V(a_2,i_2,\theta_2) - a_2 + a_1)\,(a_2 + i_2\theta_2 b) \leq (a_1 + i_1\theta_1 b)\,V(a_2,i_2,\theta_2) \\
\Leftrightarrow\quad & V(a_2,i_2,\theta_2)\,(a_2 + i_2\theta_2 b - a_1 - i_1\theta_1 b) \leq (a_2 - a_1)\,(a_2 + i_2\theta_2 b) \\
\Leftrightarrow\quad & V(a_2,i_2,\theta_2) + \tfrac{V(a_2,i_2,\theta_2)(i_2\theta_2 b - i_1\theta_1 b)}{a_2 - a_1} \leq a_2 + i_2\theta_2 b \\
\Leftrightarrow\quad & \tfrac{V(a_2,i_2,\theta_2)(i_2\theta_2 b - i_1\theta_1 b)}{a_2 - a_1} \leq -i_2 \\
\Leftrightarrow\quad & \tfrac{i_2\theta_2 - i_1\theta_1}{a_2 - a_1} b \leq -\tfrac{i_2}{V(a_2,i_2,\theta_2)}.
\end{aligned}$$

Similarly, condition (A.10) yields $\frac{i_1\theta_1 - i_2\theta_2}{a_1 - a_2} b \geq -\frac{i_1}{V(a_1, i_1, \theta_1)}$, and hence

$$-\frac{i(a_1)}{V(a_1, i(a_1), \theta(a_1))} \leq \frac{i(a_2)\theta(a_2) - i(a_1)\theta(a_1)}{a_2 - a_1} b \leq -\frac{i(a_2)}{V(a_2, i(a_2), \theta(a_2))}.$$

Since $i(a)$ and $\theta(a)$ and hence $V(a, i(a), \theta(a))$ are continuous on A, the squeeze theorem implies (A.8).

To see that (A.8) deters deviation, redefine $\Pi(a, i, \theta, p)$ by $\tilde{\pi}$, a function of firm type, transaction size, transaction surplus and the market belief about firm type:

$$\tilde{\pi}(a, i, s, \tilde{a}) = \ln \Pi\left(a, i, \frac{s}{ib}, \tilde{a} + s\right).$$

Notice the partial derivatives of $\tilde{\pi}(a, i, s, \tilde{a})$ satisfy

$$\tilde{\pi}_i(a, i, s, \tilde{a}) = \frac{1}{a + i + s} - \frac{1}{\tilde{a} + i + s},$$

$$\tilde{\pi}_s(a, i, s, \tilde{a}) = \frac{1}{\tilde{a} + s} + \frac{1}{a + i + s} - \frac{1}{\tilde{a} + i + s},$$

and

$$\tilde{\pi}_i(a, i, s, \tilde{a}) = \frac{i}{(\tilde{a} + s)(\tilde{a} + i + s)}.$$

Consider the payoff of a type-a firm from mimicking $\tilde{a}$:

$$\begin{aligned}&\frac{d}{d\tilde{a}}\tilde{\pi}(a, i(\tilde{a}), S(i(\tilde{a}), \theta(\tilde{a})), \tilde{a}) \\ &= \tilde{\pi}_i \cdot \frac{di(\tilde{a})}{d\tilde{a}} + \tilde{\pi}_s \cdot \frac{dS(i(\tilde{a}), \theta(\tilde{a}))}{d\tilde{a}} + \tilde{\pi}_{\tilde{a}}.\end{aligned} \tag{A.11}$$

Condition (A.8) implies that at $\tilde{a} = a$,

$$\begin{aligned}&\tfrac{d}{d\tilde{a}}\tilde{\pi}(a, i(\tilde{a}), S(i(\tilde{a}), \theta(\tilde{a})), \tilde{a})\,|_{\tilde{a}=a} \\ &= 0 \cdot \tfrac{di(a)}{da} + \tfrac{1}{a+S(i(a),\theta(a))}\tfrac{dS(i(a),\theta(a))}{da} + \tfrac{i}{(a+s)(a+i+s)} \\ &= 0.\end{aligned}$$

Furthermore, note that in the decomposition (A.11), $\tilde{\pi}_i$ and $\tilde{\pi}_s$ are strictly decreasing in a, and $\tilde{\pi}_{\tilde{a}}$ is invariant to a. In the issue game, according to (A.8), $\frac{dS(i(\tilde{a}),\theta(\tilde{a}))}{d\tilde{a}} < 0$. That $\frac{di(\tilde{a})}{d\tilde{a}} \leq 0$ implies $\frac{d}{d\tilde{a}}\tilde{\pi}(a, i(\tilde{a}), S(i(\tilde{a}), \theta(\tilde{a})), \tilde{a})$ increases with a. Therefore, when $\tilde{a} \gtrless a$,

$$\frac{d}{d\tilde{a}}\tilde{\pi}(a, i(\tilde{a}), S(i(\tilde{a}), \theta(\tilde{a})), \tilde{a}) \lesseqgtr 0. \tag{A.12}$$

Hence, type-a firm has no incentives to mimic any other firm in A.

In the repurchase game, that $\frac{d\theta(\tilde{a})}{d\tilde{a}} = 0$ implies

$$\frac{dS\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right)}{d\tilde{a}} = \theta b \frac{di\left(\tilde{a}\right)}{d\tilde{a}} > 0, \tag{A.13}$$

and hence

$$\begin{aligned} & \frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right), \tilde{a}\right) \\ &= \frac{1+\theta b}{a+i+s}\cdot\frac{di\left(\tilde{a}\right)}{d\tilde{a}} + C \end{aligned}$$

where C is a constant with respect to a. Since (A.13) implies $\frac{di(a)}{da} < 0$,

$$\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right), \tilde{a}\right)$$

increases with a, and hence (A.12) holds when $\tilde{a} \gtrless a$. Type-a firm has no incentives to mimic any other firm in A, completing the proof. □

Proof of Proposition 2:

Proof. Proposition 1 implies that all repurchasing firms use method $\theta = 1$ in D1 equilibria. Furthermore, Lemma 2 implies that better firms repurchase more, that is $|i(a)|$ is weakly increasing (or $i(a)$ is weakly decreasing). By the assumptions of footnote 5, all types repurchase. Therefore, there is a lower interval of firms $[a_{\min}, \hat{a})$ that repurchase the minimum size ($i = I_L$) and an upper interval of firms $(\hat{a}, a_{\max}]$ that repurchase more ($i < I_L$). One of the two intervals may be empty.

Step 1. Firms with $a > \hat{a}$ separate on different i.

Suppose in contrast, there is an equilibrium repurchase strategy $(i, 1)$ for some $i < I_L$ adopted by a non-singleton set of firms A. Consider type $\tilde{a} < E\left[a|a \in A\right]$ in A. Since

$$P(i,1) = E\left[a|a\in A\right] + S\left(i,1\right) > \tilde{a} + S\left(i,1\right),$$

the equilibrium payoff of type-$\tilde{a}$ firm satisfies

$$\Pi\left(\tilde{a}, i, 1, P(i,1)\right) < \Pi\left(\tilde{a}, i, 1, \tilde{a} + S\left(i,1\right)\right) = \tilde{a} + S\left(i,1\right).$$

Since $\lim_{\tilde{i}\downarrow i} S\left(\tilde{i},1\right) = S\left(i,1\right)$, it is possible to choose an $\tilde{i} > i$ (i.e., repurchase less $|\tilde{i}| < |i|$), such that

$$\tilde{a} + S\left(\tilde{i},1\right) > \Pi\left(\tilde{a}, i, 1, P(i,1)\right).$$

Lemma 3 implies that under D1, the price associated with the potential deviation $\left(\tilde{i},1\right)$ satisfies

$$P\left(\tilde{i},1\right) \leq \inf A + S\left(\tilde{i},1\right).$$

Type-$\tilde{a}$ firm therefore strictly prefers $(\tilde{i},1)$ to $(i,1)$:

$$\Pi\left(\tilde{a},\tilde{i},1,P(\tilde{i},1)\right) \geq \tilde{a}+S\left(\tilde{i},1\right) > \Pi\left(\tilde{a},i,1,P(i,1)\right),$$

leading to a contradiction.

Step 2. For $a > \hat{a}$, $i(a)$ satisfies (6).

Since $i(a)$ is strictly decreasing, it is sufficient to show $i(a)$ has no jump on $(\hat{a}, a_{\max}]$, which in turn implies continuity, and Lemma 4 establishes (6).

Suppose in contrast, there exists $a^* > \hat{a}$ such that

$$\bar{i} \equiv \lim_{a\uparrow a^*} i(a) > \underline{i} \equiv \lim_{a\downarrow a^*} i(a).$$

Lemma 3 implies

$$P(\underline{i},1) = a^* + S(\underline{i},1).$$

Combined with $|\underline{i}| > |\bar{i}|$, this implies

$$\Pi\left(a^*,\underline{i},1,P(\underline{i},1)\right) = a^* + S(\underline{i},1) > a^* + S(\bar{i},1).$$

Since $\Pi\left(a,\underline{i},1,P(\underline{i},1)\right)$ is continuous in a, and

$$\lim_{a\uparrow a^*} a + S(i\left(a\right),1) = a^* + S(\bar{i},1),$$

there is $a < a^*$ that benefits from deviating to $(\underline{i},1)$:

$$\Pi\left(a,\underline{i},1,P(\underline{i},1)\right) > a + S(i\left(a\right),1),$$

leading to a contradiction.

Step 3. If type $a_{\max}$ chooses $i < I_L$, it chooses $i = I_H$.

Suppose in contrast, type $a_{\max}$ chooses repurchase size $i \in (I_H, I_L)$. Step 1 implies type $a_{\max}$ is fairly priced and has payoff

$$\Pi^*(a_{\max}) = a_{\max} + S(i,1).$$

Lemma 3 implies the price associated with the deviation $(I_H, 1)$

$$P(I_H,1) = a_{\max} + S(I_H,1),$$

and hence

$$\begin{aligned}\Pi\left(a_{\max}, I_H, 1, P(I_H,1)\right) &= a_{\max} + S\left(I_H, 1\right) \\ &> a_{\max} + S\left(i, 1\right) \quad , \\ &= \Pi^*(a_{\max})\end{aligned}$$

that is type $a_{\max}$ benefits from deviating to $(I_H, 1)$, contradiction.

Step 4. If $a_{\min}$ and $a_{\max}$ are close enough such that $\hat{i}(a) < I_L$ for all a, where $\hat{i}$ is the unique solution to the ODE (6) and boundary condition (7), then all types repurchase strictly more than $|I_L|$, i.e., $\hat{a} = a_{\min}$.

Suppose in contrast, $\hat{a} > a_{\min}$. Type $a_{\min}$ prefers $\left(\hat{i}(a_{\min}), 1\right)$ at price $a_{\min} + S\left(\hat{i}(a_{\min}), 1\right)$ over $(I_L, 1)$ at the equilibrium price $E\left[a|a \in [a_{\min}, \hat{a})\right] + S\left(I_L, 1\right)$, since the latter leads to lower surplus and more unfavorable (meaning higher) market belief:

$$\begin{aligned}\Pi\left(a_{\min}, \hat{i}(a_{\min}), 1, a_{\min} + S\left(\hat{i}(a_{\min}), 1\right)\right) &= a_{\min} + S\left(\hat{i}(a_{\min}), 1\right) \\ &> a_{\min} + S\left(I_L, 1\right) \\ &= \Pi\left(a_{\min}, I_L, 1, a_{\min} + S\left(I_L, 1\right)\right) \\ &> \Pi\left(a_{\min}, I_L, 1, E\left[a|a \in [a_{\min}, \hat{a})\right] + S\left(I_L, 1\right)\right)\end{aligned} .$$

Since $\hat{i}(a_{\min}) < I_L$, Lemma 2 implies type $\hat{a}$ has the same preference. Moreover, Lemma 4 implies

$$\Pi\left(\hat{a}, \hat{i}(\hat{a}), 1, \hat{a} + S\left(\hat{i}(\hat{a}), 1\right)\right) > \Pi\left(\hat{a}, \hat{i}(a_{\min}), 1, a_{\min} + S\left(\hat{i}(a_{\min}), 1\right)\right),$$

that is type $\hat{a}$ prefers $\left(\hat{i}(\hat{a}), 1\right)$ at the equilibrium price over $\left(\hat{i}(a_{\min}), 1\right)$ at price $a_{\min}+S\left(\hat{i}(a_{\min}), 1\right)$. Therefore, type $\hat{a}$ strictly prefers $\left(\hat{i}(\hat{a}), 1\right)$ over $(I_L, 1)$. By continuity, some type $a \in (a_{\min}, \hat{a})$ has the same preference, and hence deviates to $\left(\hat{i}(\hat{a}), 1\right)$, leading to a contradiction. □

Proof of Proposition 3:

Proof. Following Propositions 1 and 2, we only need to show that the previously constructed cutoff $\hat{a}$ uniquely exists, and the conjectured equilibrium indeed satisfies D1 refinement.

We first show the uniqueness of $\hat{a}$.

Proposition 2 establishes the unique $\hat{a}$ is $a_{\min}$ when the range of $[a_{\min}, a_{\max}]$ is small enough such that $\hat{i}(a) < I_L$ for all a. Now, suppose that the range of $[a_{\min}, a_{\max}]$ is big enough such that there exists an $a_0 > a_{\min}$ such that $\hat{i}(a_0) = I_L$. This implies $\hat{a} > a_{\min}$. We will show that if there is $\hat{a}$ that satisfies (9), then $\hat{a}$ is uniquely determined by (9). If (9) has no solution, then $\hat{a} = a_{\max}$.

Step 1. (9) holds with inequality $\leq$.

Since $\hat{a} > a_{\min}$, That types below $\hat{a}$ prefer $(I_L, 1)$ over $\left(\hat{i}\left(\hat{a}\right), 1\right)$ and continuity imply

$$\Pi\left(\hat{a}, I_L, 1, P\left(I_L, 1\right)\right) \geq \Pi\left(\hat{a}, \hat{i}\left(\hat{a}\right), 1, P\left(\hat{i}\left(\hat{a}\right), 1\right)\right).$$

Step 2. If $\hat{a} < a_{\max}$, $\hat{a}$ solves (9).

That types above $\hat{a}$ prefer $\left(\hat{i}\left(a\right), 1\right)$ over $(I_L, 1)$ and continuity imply

$$\Pi\left(\hat{a}, I_L, 1, P\left(I_L, 1\right)\right) \leq \Pi\left(\hat{a}, \hat{i}\left(\hat{a}\right), 1, P\left(\hat{i}\left(\hat{a}\right), 1\right)\right).$$

Combined with step 1, this implies $\hat{a}$ solves (9).

Step 3. For $\hat{a} < a_{\max}$ and $\tilde{a} > \hat{a}$, (9) holds with inequality $>$. This implies $\breve{a}$ is unique.

Lemma 4 implies type $\tilde{a} > \hat{a}$ strictly prefers $\left(\hat{i}(\tilde{a}), 1\right)$ at price $\tilde{a} + S\left(\hat{i}(\tilde{a}), 1\right)$ to $\left(\hat{i}(\hat{a}), 1\right)$ at price $\hat{a} + S\left(\hat{i}(\hat{a}), 1\right)$. That $\hat{a} < a_{\max}$ implies $\hat{a}$ satisfies (9). Then Lemma 2 implies type $\tilde{a}$ strictly prefers $\left(\hat{i}(\hat{a}), 1\right)$ at price $\hat{a} + S\left(\hat{i}(\hat{a}), 1\right)$ to $(I_L, 1)$ at price $E\left[a|a \in [a_{\min}, \hat{a})\right] + S\left(I_L, 1\right)$. Since repurchase price $E\left[a|a \in [a_{\min}, \hat{a})\right] + S\left(I_L, 1\right)$ is more favourable (lower) than price $E\left[a|a \in [a_{\min}, \tilde{a})\right] + S\left(I_L, 1\right)$, the above chain implies (9) holds for $\tilde{a}$ with inequality $>$.

The remainder of the proof shows that under the D1 price specified in Lemma 3, no type deviates.

Step 1. No firm mimics another type.

It follows Lemma 4 that types with $a > \hat{a}$ do not mimic each other.

If $\hat{a} > a_{\min}$, (9) holds with $\leq$, which implies type $\hat{a}$ weakly prefers $(I_L, 1)$ over $\left(\hat{i}\left(\hat{a}\right), 1\right)$, and hence over $\left(\hat{i}\left(a\right), 1\right)$ for any $a \geq \hat{a}$. Lemma 2 hence implies types with $a < \hat{a}$ have the same preference and do not mimic $a \geq \hat{a}$.

If $\hat{a} < a_{\max}$, (9) holds with $\geq$, which implies type $\hat{a}$ weakly prefers $\left(\hat{i}\left(\hat{a}\right), 1\right)$ over $(I_L, 1)$. Lemma 4 implies types with $a > \hat{a}$ has the same preference, and therefore do not deviate to $(I_L, 1)$ mimicking $a < \hat{a}$.

Step 2. No firm deviates to an off-equilibrium action (i, θ) with $i\left(1 + \theta b\right) < i\left(a_{\max}\right)\left(1 + b\right)$.

Lemma 3 implies $P\left(i, \theta\right) = a_{\max} + S\left(i, \theta\right)$. Meanwhile, $P\left(I_H, 1\right) = a_{\max} + S\left(I_H, 1\right)$. Therefore, type $a_{\max}$ is fairly priced under both (i, θ) and $(I_H, 1)$. It prefers $(I_H, 1)$ since it leads to higher surplus:

$$\begin{aligned}
\Pi\left(a_{\max}, I_H, 1, P\left(I_H, 1\right)\right) &= a_{\max} + S\left(I_H, 1\right) \\
&> a_{\max} + S\left(i, \theta\right) \\
&= \Pi\left(a_{\max}, i, \theta, P\left(i, \theta\right)\right).
\end{aligned}$$

If $\hat{a} < a_{\max}$, $i(a_{\max}) = I_H$. If $\hat{a} = a_{\max}$, $i(a_{\max}) = I_L$ or I_H, and (9) holds with $\leq$, implying $a_{\max}$ weakly prefers $(i(a_{\max}), 1)$ to $(I_H, 1)$. In both cases, type $a_{\max}$ prefers its equilibrium choice $(i(a_{\max}), 1)$ to (i, θ). Lemma 2 then implies all types have the same preference. Since no type mimics type $a_{\max}$ according to Step 1, no type deviates to (i, θ).

Step 3. No firm deviates to an off-equilibrium action (i, θ) with

$$i(1+\theta b) \in \left[i(a_{\max})(1+b), \hat{i}(\hat{a})(1+b)\right]. \tag{A.14}$$

Condition (A.14) and the continuity of $\hat{i}(\cdot)$ imply that there is $\bar{a} \in [\hat{a}, a_{\max}]$ such that

$$i(1+\theta b) = \hat{i}(\bar{a})(1+b). \tag{A.15}$$

Since $\theta \leq 1$, condition (A.15) implies that $|i| < |\hat{i}(\bar{a})|$, which in turn implies that $S(i, \theta) < S\left(\hat{i}(\bar{a}), 1\right)$. Lemma 3 implies $P(i, \theta) = \bar{a} + S(i, \theta)$. Type $\bar{a}$ is fairly priced under both (i, θ) and $\left(\hat{i}(\bar{a}), 1\right)$, and therefore it weakly prefers $\left(\hat{i}(\bar{a}), 1\right)$ over (i, θ). Lemma 2 implies all types weakly prefer $\left(\hat{i}(\bar{a}), 1\right)$ over (i, θ). Since no type deviates to $\left(\hat{i}(\bar{a}), 1\right)$ according to Step 1, no type deviates to (i, θ).

Step 4. No firm deviates to an off-equilibrium action (i, θ) with $i(1+\theta b) > \hat{i}(\hat{a})(1+b)$.

Repeating the analysis in Step 3, we have $|i| < |\hat{i}(\bar{a})|$ and $i\theta b < \hat{i}(\hat{a})b$. Lemma 3 implies $P(i, \theta) = \hat{a} + S(i, \theta)$. Hence, type $\hat{a}$ is fairly priced under both choices (i, θ) and $\left(\hat{i}(\hat{a}), 1\right)$, with the latter generating higher surplus. Therefore, type $\hat{a}$ prefers $\left(\hat{i}(\hat{a}), 1\right)$ to (i, θ). Lemma 2 implies types $a > \hat{a}$ have the same preference. Since they do not benefit from mimicking type $\hat{a}$, they do not deviate to (i, θ). If $\hat{a} > a_{\min}$, then (9) holds with $\leq$. Type $\hat{a}$ weakly prefers $(I_L, 1)$ to $\left(\hat{i}(\hat{a}), 1\right)$, and hence to (i, θ). Since $I_L(1+b) > i(1+\theta b)$, Lemma 2 implies types $a < \hat{a}$ also prefer $(I_L, 1)$ to (i, θ), and hence do not deviate to (i, θ). $\square$

Proof of Proposition 4:

Proof. Suppose in contrast, in a D1 equilibrium of the issue game, firm types in set A choose (i, θ) such that $i > I_L$ and $\theta < 1$. Consider firm $\tilde{a} \in A$ with $\tilde{a} \geq E[a | a \in A]$. This implies

$$P(i, \theta) \leq \tilde{a} + S(i, \theta).$$

We can choose $\tilde{i} < i$ and a corresponding $\tilde{\theta} \in \left(\frac{i\theta}{\tilde{i}}, \left(\frac{i(1+\theta b)}{\tilde{i}} - 1\right) b^{-1}\right)$, which imply that $\tilde{i}(1+\tilde{\theta}b) < i(1+\theta b)$ and $\tilde{i}\tilde{\theta}b > i\theta b$. Lemma 3 then implies

$$P\left(\tilde{i}, \tilde{\theta}\right) \geq \tilde{a} + S\left(\tilde{i}, \tilde{\theta}\right) > \tilde{a} + S(i, \theta).$$

This implies $\left(\tilde{i}, \tilde{\theta}\right)$ leads to higher issue surplus and better issue price than (i, θ). Then firm $\tilde{a}$ strictly prefers $(\tilde{i}, \tilde{\theta})$ to (i, θ):

$$\begin{aligned}
\Pi\left(\tilde{a}, \tilde{i}, \tilde{\theta}, P(\tilde{i}, \tilde{\theta})\right) &> \Pi\left(\tilde{a}, \tilde{i}, \tilde{\theta}, \tilde{a} + S\left(\tilde{i}, \tilde{\theta}\right)\right) \\
&= \tilde{a} + S\left(\tilde{i}, \tilde{\theta}\right) \\
&> \tilde{a} + S\left(i, \theta\right) \\
&= \Pi\left(\tilde{a}, i, \theta, \tilde{a} + S\left(i, \theta\right)\right) \\
&\geq \Pi\left(\tilde{a}, i, \theta, P\left(i, \theta\right)\right)
\end{aligned}.$$

This contradiction completes the proof. □

Proof of Proposition 5:

Proof. We first show the uniqueness of the D1 equilibrium.

Proposition 4 guarantees a firm either chooses $(i, 1)$ for some i or (I_L, θ) for some θ. Lemma 2 implies firms' choice of i, denoted by $i(a)$ for type a, and choice of θ, denoted by $\theta(a)$ for type a, are weakly decreasing in a. Therefore, there is a cutoff $\hat{a}$ such that types with $a > \hat{a}$ choose $(i, 1)$ for some i and types with $a < \hat{a}$ choose (I_L, θ) for some θ.

Step 1. No type chooses $(I_L, 0)$ in equilibrium.

The assumptions of footnote 5 imply that type $a_{\max}$ strictly prefers $(I_L, 1)$ to $(0, 1)$ (doing nothing) under any prices. On the other hand, type $a_{\max}$ prefers $(0, 1)$ over $(I_L, 0)$ under any prices, the latter of which implies zero surplus but weak under-pricing:

$$\begin{aligned}
\Pi\left(a_{\max}, 0, 1, P\left(0, 1\right)\right) &= a_{\max} \\
&= a_{\max} + S\left(I_L, 0\right) \\
&= \Pi\left(a_{\max}, I_L, 0, a_{\max} + S\left(I_L, 0\right)\right) \\
&\geq \Pi\left(a_{\max}, I_L, 0, P\left(I_L, 0\right)\right)
\end{aligned}.$$

Therefore, type $a_{\max}$ strictly prefers $(I_L, 1)$ to $(I_L, 0)$. According to Lemma 2, all types have the same preference. Therefore, no type chooses $(I_L, 0)$ in equilibrium.

Step 2. All types separate on different pairs of (i, θ).

Suppose in contrast, types in set A pool on (i, θ) in equilibrium.

$$P\left(i, \theta\right) = E\left[a | a \in A\right] + S\left(i, \theta\right).$$

According to step 1, $(i,\theta) \neq (I_L, 0)$. There is $\left(\tilde{i},\tilde{\theta}\right)$ such that $\tilde{i}\left(1+\tilde{\theta}b\right) < i(1+\theta b)$. Lemma 3 implies

$$P^*\left(\tilde{i},\tilde{\theta}\right) \geq \sup A + S\left(\tilde{i},\tilde{\theta}\right).$$

Types in A benefit from deviating to $\left(\tilde{i},\tilde{\theta}\right)$ that are close enough to (i,θ), because it leads to marginal changes in i and θ but a discrete improvement in market belief. This leads to a contradiction.

Step 3. $i(a)$ and $\theta(a)$ are continuous.

Given $i(a)$ and $\theta(a)$ are decreasing, it is sufficient to show there is no jump. Suppose in contrast, there is a^* such that

$$\lim_{a\uparrow a^*} i(a) > \lim_{a\downarrow a^*} i(a)$$

or

$$\lim_{a\uparrow a^*} \theta(a) > \lim_{a\downarrow a^*} \theta(a).$$

Let $\bar{i} \equiv \lim_{a\uparrow a^*} i(a)$, $\underline{i} \equiv \lim_{a\downarrow a^*} i(a)$, $\bar{\theta} \equiv \lim_{a\uparrow a^*} \theta(a)$, $\underline{\theta} \equiv \lim_{a\downarrow a^*} \theta(a)$. Since all types fully separate, as a approaches a^* from above, their equilibrium payoff, $a + S(i(a), \theta(a))$, approaches $a^* + S(\underline{i},\underline{\theta})$:

$$\lim_{a\downarrow a^*} \Pi^*(a) = \lim_{a\downarrow a^*} [a + S(i(a), \theta(a))] = a^* + S(\underline{i},\underline{\theta}).$$

Lemma 3 implies

$$P\left(\bar{i},\bar{\theta}\right) = a^* + S\left(\bar{i},\bar{\theta}\right).$$

As a approaches a^* from above, their payoff from deviating to $\left(\bar{i},\bar{\theta}\right)$ approaches $a^* + S\left(\bar{i},\bar{\theta}\right)$:

$$\lim_{a\downarrow a^*} \Pi\left(a,\bar{i},\bar{\theta},P\left(\bar{i},\bar{\theta}\right)\right) = \Pi\left(a^*,\bar{i},\bar{\theta},P\left(\bar{i},\bar{\theta}\right)\right) = a^* + S\left(\bar{i},\bar{\theta}\right).$$

Since $a^* + S(\underline{i},\underline{\theta}) < a^* + S\left(\bar{i},\bar{\theta}\right)$, there $a > a^*$ such that

$$\Pi^*(a) < \Pi\left(a,\bar{i},\bar{\theta},P\left(\bar{i},\bar{\theta}\right)\right),$$

that is type a deviates to $\left(\bar{i},\bar{\theta}\right)$. This leads to a contradiction.

Step 4. Lemma 4 implies for $a > \hat{a}$, $i(a)$ is differentiable and satisfies ODE (6); for $a < \hat{a}$, $\theta(a)$ is differentiable and satisfies ODE (15)

Step 5. Type $a_{\min}$ chooses $(I_H, 1)$.

Suppose in contrast, $a_{\min}$ chooses $(i,\theta) \neq (I_H, 1)$. This implies $i\theta < I_H$. Type $a_{\min}$'s equilibrium payoff is $a_{\min} + S(i,\theta)$. If type $a_{\min}$ deviates to $(I_H, 1)$, it receives higher surplus and no worse market belief:

$$\begin{aligned}\Pi\left(a_{\min}, I_H, 1, P\left(I_H, 1\right)\right) &\geq \Pi\left(a_{\min}, I_H, 1, a_{\min} + S\left(I_H, 1\right)\right) \\ &= a_{\min} + S\left(I_H, 1\right) \\ &> a_{\min} + S\left(i, \theta\right).\end{aligned}$$

Therefore, type $a_{\min}$ deviates to $(I_H, 1)$, leading to a contradiction.

Next, we show the existence of the conjectured equilibrium under the D1 price as specified in Lemma 3.

Step 1. There is enough space for all types to separate according to $\hat{\imath}(\cdot)$ and $\hat{\theta}(\cdot)$.

Suppose in contrast, there is $a > \hat{a}$ such that $\hat{\theta}(a) = 0$. Lemma 4 implies type a strictly prefers $(I_L, 0)$ under belief a over $(I_L, 1)$ under belief $\hat{a}$.

On the other hand, under the assumptions of footnote 5, type $a_{\max}$ prefers $(I_L, 1)$ under the worst belief, $a_{\min}$, to doing nothing, the latter of which leads to payoff $a_{\max}$, and hence is equivalent to choosing $(I_L, 0)$ under the correct belief $a_{\max}$. This implies type $a_{\max}$also prefers $(I_L, 1)$ under belief $\hat{a}$ to $(I_L, 0)$ under belief $a_{\max}$. Lemma 2 implies all types have the same preference. Therefore, type a prefers $(I_L, 1)$ under belief $\hat{a}$ to $(I_L, 0)$ under belief a. This leads to a contradiction.

Step 2. Lemma 4 implies no type benefits from mimicking another type.

Step 3. Under a D1 belief, no type benefits from deviating to an off-equilibrium action (i, θ) that satisfies

$$i\left(1 + \theta b\right) > \hat{\imath}\left(a_{\max}\right)\left(1 + \hat{\theta}\left(a_{\max}\right) b\right). \tag{A.16}$$

(A.16) implies there is a^* such that

$$i\left(1 + \theta b\right) = \hat{\imath}\left(a^*\right)\left(1 + \hat{\theta}\left(a^*\right) b\right).$$

Lemma 3 implies

$$P\left(i, \theta\right) = a^* + S\left(i, \theta\right).$$

Since (i, θ) is off-equilibrium, $i > \hat{\imath}(a^*)$, which implies $i\theta b < \hat{\imath}(a^*)\,\hat{\theta}(a^*)\,b$. Since type a^* is fairly priced under both (i, θ) and its equilibrium choice $\left(\hat{\imath}(a^*), \hat{\theta}(a^*)\right)$, it prefers $\left(\hat{\imath}(a^*), \hat{\theta}(a^*)\right)$, which leads to higher surplus:

$$\begin{aligned}\Pi\left(a^*, \hat{\imath}\left(a^*\right), \hat{\theta}\left(a^*\right), P\left(\hat{\imath}\left(a^*\right), \hat{\theta}\left(a^*\right)\right)\right) &= a^* + S\left(\hat{\imath}\left(a^*\right), \hat{\theta}\left(a^*\right)\right) \\ &> a^* + S\left(i, \theta\right) \\ &= \Pi\left(a^*, i, \theta, P\left(i, \theta\right)\right).\end{aligned}$$

Lemma 2 implies all types have the same preference. By Lemma 4, no type benefits from deviating to (i, θ).

Step 4. No type benefits from deviating to an off-equilibrium action (i, θ) that satisfies

$$i\left(1+\theta b\right) \leq \hat{i}\left(a_{\max}\right)\left(1+\hat{\theta}\left(a_{\max}\right) b\right). \tag{A.17}$$

Lemma 3 implies

$$P\left(i, \theta\right) = a_{\max} + S\left(i, \theta\right).$$

(A.17) implies $i\theta < \hat{i}\left(a_{\max}\right) \hat{\theta}\left(a_{\max}\right)$. Therefore, type $a_{\max}$ prefers $\left(\hat{i}\left(a_{\max}\right), \hat{\theta}\left(a_{\max}\right)\right)$ over (i, θ) since the latter leads to lower surplus:

$$\begin{aligned} \Pi\left(a_{\max}, i, \theta, P\left(i, \theta\right)\right) &= a_{\max} + S\left(i, \theta\right) \\ &< a_{\max} + S\left(\hat{i}\left(a_{\max}\right), \hat{\theta}\left(a_{\max}\right)\right) \\ &= \Pi\left(a_{\max}, \hat{i}\left(a_{\max}\right), \hat{\theta}\left(a_{\max}\right), P\left(\hat{i}\left(a_{\max}\right), \hat{\theta}\left(a_{\max}\right)\right)\right) \end{aligned}.$$

Lemma 2 implies all types have the same preference. Lemma 4 implies no type benefits from deviating to (i, θ). This completes the proof. □

Proof of Lemma 1:

Proof. Notice

$$\pi\left(a, i, \theta, \tilde{a}\right) = \ln V\left(a, i, \theta\right) - \ln\left(1 + \frac{i}{\tilde{a} + S\left(i, \theta\right)}\right).$$

So

$$\pi_a\left(a, i, \theta, \tilde{a}\right) = \frac{V_a\left(a, i, \theta\right)}{V\left(a, i, \theta\right)} = \frac{1}{V\left(a, i, \theta\right)},$$

and

$$\pi_{ai} = \frac{-V_i\left(a, i, \theta\right)}{V\left(a, i, \theta\right)^2}.$$

Moreover,

$$\begin{aligned} \pi_i\left(a, i, \theta, a\right) &= \frac{\partial \ln V\left(a, i, \theta\right)}{\partial i} - \frac{\partial \ln\left(1 + \frac{i}{a + S(i,\theta)}\right)}{\partial i} \\ &= \frac{\partial\left[\ln V\left(a, i, \theta\right) - \ln\left(1 + \frac{i}{a + S(i,\theta)}\right)\right]}{\partial i} \\ &= \frac{\partial \ln\left(a + S\left(i, \theta\right)\right)}{\partial i} \\ &= \frac{S_i\left(i, \theta\right)}{a + S\left(i, \theta\right)}. \end{aligned}$$

Therefore,

$$\frac{-\pi_{ai}\left(a, i, \theta, a\right)}{\pi_i\left(a, i, \theta, a\right)} = \frac{V_i\left(a, i, \theta\right)}{S_i\left(a, i, \theta\right)} \cdot \frac{a + S\left(i, \theta\right)}{V\left(a, i, \theta\right)^2}.$$

Similarly,

$$\frac{-\pi_{a\theta}(a,i,\theta,a)}{\pi_{\theta}(a,i,\theta,a)}=\frac{V_{\theta}(a,i,\theta)}{S_{\theta}(a,i,\theta)}\cdot\frac{a+S(i,\theta)}{V(a,i,\theta)^{2}}.$$

Since $\frac{a+S(i,\theta)}{V(a,i,\theta)^{2}}>0$,

$$\frac{-\pi_{ai}(a,i,\theta,a)}{\pi_{i}(a,i,\theta,a)}-\frac{-\pi_{a\theta}(a,i,\theta,a)}{\pi_{\theta}(a,i,\theta,a)}$$

has the same sign as

$$\frac{V_{i}(a,i,\theta)}{S_{i}(a,i,\theta)}-\frac{V_{\theta}(a,i,\theta)}{S_{\theta}(a,i,\theta)}.$$

□

Online Appendix

B Optionality in Smooth Methods

We show that the equilibrium outcomes remain unchanged in the modification of the model outlined in footnote 8, which incorporates firms' option to privately choose the actual issue or repurchase size i^A with the announced size i being an upper bound. In brief, firms choose to issue or repurchase the full size that is announced, $i^A = i$, and the unique D1 equilibria of the repurchase game and the issue game are still those described in Proposition 3 and 5.

Redefine a "D1 belief" by substituting (5) with

$$D_a(i,\theta) = \left\{p : \Pi\left(a, i^A, \theta, p\right) > \Pi^*(a) \quad \exists \left|i^A\right| \in \left[\left|I_L\right|, \left|i\right|\right]\right\}.$$

D1 hence requires the belief associated with an announced size i and efficiency θ only be supported on types a that benefit from these choices (combined with any private choice $\left|i^A\right| \in \left[\left|I_L\right|, \left|i\right|\right]$) under the largest set of prices.

Lemma 5. *If*

$$\Pi\left(a, i^A, \theta, p\right) = \frac{a + i^A(1+\theta b)}{1 + \frac{i^A}{p}} > a, \tag{B.1}$$

then $\Pi\left(a, i^A, \theta, p\right)$ *strictly increases with* $\left|i^A\right|$.

Proof. In the issue game, (B.1) is equivalent to

$$a - (1+\theta b)\, p < 0,$$

which implies

$$\Pi\left(a, i^A, \theta, p\right) = (1+\theta b)\, p + \frac{a - (1+\theta b)\, p}{1 + \frac{i^A}{p}} \tag{B.2}$$

increases with i^A. In the repurchase game, (B.1) is equivalent to

$$a - (1+\theta b)\, p > 0.$$

Since $i^A < 0$, this implies (B.2) decreases with i^A, that is, increases with $\left|i^A\right|$. □

By Lemma 5, firms' equilibrium choices should satisfy $i^A = i$. Moreover, Lemma 5 implies in any equilibrium or conjectured equilibrium, $D_a(i,\theta)$ is the same as that defined in the baseline model (5), and hence the set of beliefs that satisfy D1 is the same as in the baseline model.

Then it follows that Proposition 3 and 5 continue to hold: Conjecture an equilibrium with an arbitrary equilibrium strategy that satisfies $i = i^A$ and with a D1 belief. Type a benefits from deviating to a public choice $\left(\tilde{i}, \tilde{\theta}\right)$ with any private choice $\left|i^A\right| \in [|I_L|, |i|]$ if and only if it benefits from choosing $\left(\tilde{i}, \tilde{\theta}\right)$ with $i^A = i$. This is equivalent to that in the corresponding conjectured equilibrium of the baseline model, type a benefits from deviating to $\left(\tilde{i}, \tilde{\theta}\right)$. Therefore,

Proposition 10. *A strategy $\left(i\left(a\right), \theta\left(a\right), i^A\left(a\right)\right)$ supports a D1 equilibrium of the modified issue or repurchase game with optionality if and only if $i^A\left(a\right) = i\left(a\right)$ and $\left(i\left(a\right), \theta\left(a\right)\right)$ supports a D1 equilibrium of the baseline model.*

C Proofs of Section 5

Lemma 6. *Lemma 2 and 3 generalizes to the firm's objective function (21).*

Proof. The proofs are the same as when $\epsilon = 0$, except that (A.1) is substituted by

$$\ln \Pi(a, i_1, \theta_1, p_1) - \ln \Pi(a, i_2, \theta_2, p_2) = \epsilon \ln \frac{p_1}{p_2} + (1-\epsilon)\left(\ln \frac{V(a, i_1, \theta_1)}{V(a, i_2, \theta_2)} - \ln \frac{1+\frac{i_1}{p_1}}{1+\frac{i_2}{p_2}}\right).$$

□

Lemma 7. *In the issue or repurchase game with $\epsilon \in (0,1)$, type-a firm's overall preference for price satisfies $\frac{\partial \Pi(a,i,\theta,p)}{\partial p} \gtrless 0$ if $\epsilon p + i \gtrless 0$. In particular,*

1. *In the issue game with $\epsilon \in (0,1)$, or in the repurchase game with $\epsilon > \frac{-I_H}{a_{\min}}$, $\Pi\left(a, i, \theta, p\right)$ strictly increases with p for any a, i, θ, p;*
2. *In a D1 equilibrium of the repurchase game, $\epsilon P(i, \theta) + i \leq 0$ for $i > I_H$ such that (i, θ) is chosen by some firm.*

Proof. Notice

$$\frac{\partial \Pi\left(a, i, \theta, p\right)}{\partial p}) = \frac{\Pi}{p(p+i)}\left(\epsilon p + i\right).$$

In the issue game with $\epsilon \in (0,1)$, or in the repurchase game with $\epsilon > \frac{-I_H}{a_{\min}}$, $\epsilon p + i > 0$ for any p and i, which implies $\frac{\partial \Pi(a,i,\theta,p)}{\partial p} > 0$.

In a D1 equilibrium of the repurchase game, suppose a set A of firm types choose (i, θ) with $i > I_H$ in a D1 equilibrium. Conversely, suppose $\epsilon P(i, \theta) + i > 0$. Let $a^* = E[a|a \in A]$.

$$\Pi(a^*, i, \theta, P(i, \theta)) = P(i, \theta) = a^* + S(i, \theta).$$

By continuity, there is $\tilde{i} < i$ such that

$$\epsilon\left[a^* + S(\tilde{i},\theta)\right] + \tilde{i} > 0.$$

Since $\tilde{i} + S\left(\tilde{i},\theta\right) < i + S\left(i,\theta\right)$, Lemma 6 implies

$$P(\tilde{i},\theta) \geq a^* + S(\tilde{i},\theta).$$

Therefore, for $p \in \left[a^* + S(\tilde{i},\theta), P\left(\tilde{i},\theta\right)\right]$,

$$\epsilon p + \tilde{i} > 0$$

and hence

$$\frac{\partial}{\partial p}\Pi(a,\tilde{i},\theta,p) > 0.$$

Therefore,

$$\begin{aligned}\Pi(a^*,\tilde{i},\theta,P(\tilde{i},\theta)) &\geq \Pi(a^*,\tilde{i},\theta,a^*+S(\tilde{i},\theta))\\ &= a^* + S(\tilde{i},\theta)\\ &> a^* + S(i,\theta)\\ &= \Pi(a^*,i,\theta,P(i,\theta))\end{aligned}.$$

The second inequality is due to $S\left(\tilde{i},\theta\right) > S\left(i,\theta\right)$. This implies type a^* strictly prefers $(\tilde{i},\theta)$ to (i,θ). Lemma 6 implies there is some type in A that strictly prefers $(\tilde{i},\theta)$ to (i,θ), which leads to contradiction. □

Lemma 8. *In an equilibrium of the issue or repurchase game, if there is an interval of firm types A on which the size and method choices $i(a)$ and $\theta(a)$ are continuous, and the choices fully reveal firm types, then*

$$\frac{d\left(i(a)\theta(a)b\right)}{da} = -\frac{i(a) + \epsilon\left(a + i(a)\theta(a)b\right)}{V\left(a,i(a),\theta(a)\right)}. \tag{C.1}$$

Conversely, if on an interval of firm types A, (C.1) holds, price is fully revealing, and one of the following holds:

1. $\frac{di(a)}{da} \leq 0$ *for all $a \in A$ and this is an issue game,*
2. $\frac{d\theta(a)}{da} = 0$ *for all $a \in A$,*
$$\lim_{a\uparrow\sup A} i(a) + \epsilon\left(a + i(a)\theta(a)b\right) < 0, \tag{C.2}$$
and this is a repurchase game, or
3. $\frac{di(a)}{da} = 0$ *for all $a \in A$ and this is a repurchase game with $\epsilon > \frac{-I_H}{a_{\min}}$,*

then no type in A has an incentive to mimic another type in A.

Proof. In an equilibrium, let A be an interval of firm types on which $i(a)$ and $\theta(a)$ are continuous and price is fully revealing. This implies for all $a \in A$,

$$\Pi^*(a) = a + S\left(i(a), \theta(a)\right).$$

Consider any two firm types $a_1, a_2 \in A$ such that $[a_1, a_2] \subset A$. Equilibrium conditions imply

$$\Pi\left(a_1, i(a_2), \theta(a_2), P\left(i(a_2), \theta(a_2)\right)\right) \leq \Pi^*(a_1), \tag{C.3}$$

$$\Pi\left(a_2, i(a_1), \theta(a_1), P\left(i(a_1), \theta(a_1)\right)\right) \leq \Pi^*(a_2). \tag{C.4}$$

Using the functional form of Π and V, condition (C.3) can be explicitly written as

$$\frac{\left(a_2 + i\left(a_2\right)\theta\left(a_2\right)b\right)^{\frac{1}{1-\epsilon}} - \left(a_1 + i\left(a_1\right)\theta\left(a_1\right)b\right)^{\frac{1}{1-\epsilon}}}{a_2 - a_1} \leq \frac{\left(a_2 + i\left(a_2\right)\theta\left(a_2\right)b\right)^{\frac{1}{1-\epsilon}}}{V\left(a_2, i\left(a_2\right), \theta\left(a_2\right)\right)}$$

Similarly, condition (C.4) yields

$$\frac{\left(a_2 + i\left(a_2\right)\theta\left(a_2\right)b\right)^{\frac{1}{1-\epsilon}} - \left(a_1 + i\left(a_1\right)\theta\left(a_1\right)b\right)^{\frac{1}{1-\epsilon}}}{a_2 - a_1} \geq \frac{\left(a_1 + i\left(a_1\right)\theta\left(a_1\right)b\right)^{\frac{1}{1-\epsilon}}}{V\left(a_1, i\left(a_1\right), \theta\left(a_1\right)\right)}.$$

Since $i(a)$ and $\theta(a)$ and hence $V\left(a, i(a), \theta(a)\right)$ are continuous on A, the squeeze theorem implies (C.1).

To see that (C.1) deters deviation, redefine $\Pi\left(a, i, \theta, p\right)$ by $\tilde{\pi}$, a function of firm type, transaction size, transaction surplus and the market belief about firm type:

$$\tilde{\pi}\left(a, i, s, \tilde{a}\right) = \ln \Pi\left(a, i, \frac{s}{ib}, \tilde{a} + s\right).$$

Notice the partial derivatives of $\tilde{\pi}\left(a, i, s, \tilde{a}\right)$ satisfy

$$\tilde{\pi}_i\left(a, i, s, \tilde{a}\right) = (1-\epsilon)\left(\frac{1}{a+i+s} - \frac{1}{\tilde{a}+i+s}\right),$$

$$\tilde{\pi}_s\left(a, i, s, \tilde{a}\right) = \frac{1}{\tilde{a}+s} + (1-\epsilon)\left(\frac{1}{a+i+s} - \frac{1}{\tilde{a}+i+s}\right),$$

and

$$\tilde{\pi}_{\tilde{a}}\left(a, i, s, \tilde{a}\right) = \frac{\epsilon\left(\tilde{a}+s\right)+i}{\left(\tilde{a}+s\right)\left(\tilde{a}+i+s\right)}.$$

Consider the payoff of a type-a firm from mimicking $\tilde{a}$:

$$\begin{aligned} &\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right), \tilde{a}\right) \\ &= \tilde{\pi}_i \cdot \frac{di\left(\tilde{a}\right)}{d\tilde{a}} + \tilde{\pi}_s \cdot \frac{dS\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right)}{d\tilde{a}} + \tilde{\pi}_{\tilde{a}}. \end{aligned} \tag{C.5}$$

Condition (C.1) implies that at $\tilde{a}=a$,

$$\begin{aligned}&\tfrac{d}{d\tilde{a}}\tilde{\pi}\left(a,i\left(\tilde{a}\right),S\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right),\tilde{a}\right)|_{\tilde{a}=a}\\&=0\cdot\tfrac{di(a)}{da}+\tfrac{1}{a+S(i(a),\theta(a))}\tfrac{dS(i(a),\theta(a))}{da}+\tfrac{\epsilon[\tilde{a}+S(i(a),\theta(a))]+i}{[a+S(i(a),\theta(a))][a+i+S(i(a),\theta(a))]}\\&=0.\end{aligned}$$

Furthermore, note that in the decomposition (C.5), $\tilde{\pi}_i$ and $\tilde{\pi}_s$ are strictly decreasing in a, and $\tilde{\pi}_{\tilde{a}}$ is invariant to a.

In the issue game, according to (C.1), $\frac{dS(i(\tilde{a}),\theta(\tilde{a}))}{d\tilde{a}}<0$. That $\frac{di(\tilde{a})}{d\tilde{a}}\leq 0$ implies

$$\frac{d}{d\tilde{a}}\tilde{\pi}\left(a,i\left(\tilde{a}\right),S\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right),\tilde{a}\right)$$

increases with a. Therefore, when $\tilde{a}\gtrless a$,

$$\frac{d}{d\tilde{a}}\tilde{\pi}\left(a,i\left(\tilde{a}\right),S\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right),\tilde{a}\right)\lesseqgtr 0. \tag{C.6}$$

Hence, type-a firm has no incentives to mimic any other firm in A.

In the repurchase game,

1. If $\frac{d\theta(\tilde{a})}{d\tilde{a}}=0$ for all $\tilde{a}\in A$ and (C.2) holds, then (A.8) implies for all $\tilde{a}\in A$,

$$\frac{dS\left(i\left(\tilde{a}\right),\theta\left(\tilde{a}\right)\right)}{d\tilde{a}}=\theta b\frac{di\left(\tilde{a}\right)}{d\tilde{a}}>0 \tag{C.7}$$

and

$$\frac{di\left(\tilde{a}\right)}{d\tilde{a}}<0. \tag{C.8}$$

To see why the inequalities hold not only for $\tilde{a}$ close to $\sup A$ but also for all $\tilde{a}\in A$, suppose conversely, there is $a^*\neq\sup A$ from A such that $\theta b\frac{di(a^*)}{da^*}\leq 0$ so that $\frac{di(a^*)}{da^*}\geq 0$. Then there is $\bar{a}\in A$ such that $\frac{di(\bar{a})}{d\bar{a}}=0$ and $\frac{d^2i(\bar{a})}{d\bar{a}^2}\leq 0$. Plugging $\frac{d\theta(a)}{da}=0$ into (C.1) leads to

$$\frac{di\left(a\right)}{da}=-\frac{\left(1-\epsilon\right)i}{V\left(a,i\left(a\right),\theta\left(a\right)\right)b}-\frac{\epsilon}{b}.$$

Taking the derivative w.r.t. a leads to

$$\frac{d^2i\left(a\right)}{da^2}=\frac{1-\epsilon}{V\left(a,i\left(a\right),\theta\left(a\right)\right)^2b}\left(i(a)-a\frac{di\left(a\right)}{da}\right).$$

That $\frac{di(\bar{a})}{d\bar{a}}=0$ implies $\frac{d^2i(\bar{a})}{d\bar{a}^2}>0$, which leads to a contradiction. Therefore, (C.7) and (C.8) hold for all $\tilde{a}\in A$.

(C.7) implies

$$\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right), \tilde{a}\right)$$
$$= (1-\epsilon)\frac{1+\theta b}{a+i+s}\cdot\frac{di\left(\tilde{a}\right)}{d\tilde{a}} + C$$

where C is a constant with respect to a. (C.8) implies $\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right), \tilde{a}\right)$ increases with a, and hence (C.6) holds when $\tilde{a} \gtreqless a$. Type-a firm has no incentives to mimic any other firm in A.

2. If $\frac{di(a)}{da} = 0$ and $\epsilon > \frac{-I_H}{a_{\min}}$, then

$$\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right), \tilde{a}\right)$$
$$= (1-\epsilon)\frac{1}{a+i+s}\cdot\frac{dS\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right)}{d\tilde{a}} + C$$

where C is a constant with respect to a. That $\epsilon > \frac{-I_H}{a_{\min}}$ and (C.1) implies $\frac{dS(i(\tilde{a}),\theta(\tilde{a}))}{d\tilde{a}} < 0$. Therefore, $\frac{d}{d\tilde{a}}\tilde{\pi}\left(a, i\left(\tilde{a}\right), S\left(i\left(\tilde{a}\right), \theta\left(\tilde{a}\right)\right), \tilde{a}\right)$ increases with a, and hence (A.12) holds when $\tilde{a} \gtreqless a$. Type-a firm has no incentives to mimic any other firm in A, completing the proof.

□

Proof of Proposition 6:

Notice that $\epsilon > \frac{-I_H}{a_{\min}}$ implies $\epsilon P(i,\theta) + i > 0$ for all (i,θ), and Lemma 7 hence implies $\Pi\left(a, i, \theta, p\right)$ strictly increases with p.

We first prove the uniqueness of the D1 equilibrium.

Notice that in a D1 equilibrium, all firms choose repurchase size I_H. That $\epsilon > \frac{-I_H}{a_{\min}}$ implies $\epsilon P(i,\theta) + i > 0$ for all (i,θ), and Lemma 7 implies all repurchasing firm types repurchase amount I_H. Moreover, all firms repurchase. Suppose in contrast, firms in set A choose to do nothing, i.e., $i = 0$. Then their share price is $E\left[a|a \in A\right]$. Consider firm $\tilde{a} \in A$ such that $\tilde{a} \geq E\left[a|a \in A\right]$. Then $\tilde{a}$ benefits from deviating to $(I_L, 1)$:

$$\begin{aligned}
\Pi\left(\tilde{a}, I_L, 1, P\left(I_L, 1\right)\right) &\geq \Pi\left(\tilde{a}, I_L, 1, \tilde{a} + S\left(I_L, 1\right)\right) \\
&= \tilde{a} + S\left(I_L, 1\right) \\
&> \tilde{a} \\
&= \Pi\left(\tilde{a}, 0, \theta, \tilde{a}\right) \\
&\geq \Pi\left(\tilde{a}, 0, \theta, E\left[a|a \in A\right]\right) \\
&= \Pi^*\left(\tilde{a}\right).
\end{aligned}$$

The first inequality is because $I_L(1+b) < 0$ and hence Lemma 6 implies $P(I_L, 1) \geq \tilde{a} + S(I_L, 1)$.

Lemma 6 implies in a D1 equilibrium, $\theta(a)$ weakly decreases with a. Therefore, there is $\hat{a}$ such that firms $a \in [a_{\min}, \hat{a})$ use methods $\theta(a) > 0$ and firms $a \in (\hat{a}, a_{\max}]$ use method $\theta(a) = 0$.

Step 1. Firms with $a < \hat{a}$ separate on different θ.

Suppose in contrast, there is an equilibrium repurchase strategy (I_H, θ) for some $\theta > 0$ adopted by a non-singleton set of firms A. Consider type $\tilde{a} > E[a|a \in A]$ in A. Since

$$P(I_H, \theta) = E[a|a \in A] + S(I_H, \theta) < \tilde{a} + S(I_H, \theta),$$

the equilibrium payoff of type-$\tilde{a}$ firm satisfies

$$\Pi(\tilde{a}, I_H, \theta, P(I_H, \theta)) < \tilde{a} + S(I_H, \theta).$$

Since $\lim_{\tilde{\theta} \uparrow \theta} S\left(I_H, \tilde{\theta}\right) = S(I_H, \theta)$, it is possible to choose a $\tilde{\theta} < \theta$ such that

$$\tilde{a} + S\left(I_H, \tilde{\theta}\right) > \Pi(\tilde{a}, I_H, \theta, P(I_H, \theta)).$$

Lemma 3 implies that the price associated with the potential deviation $\left(I_H, \tilde{\theta}\right)$ satisfies

$$P\left(I_H, \tilde{\theta}\right) \geq \sup A + S\left(I_H, \tilde{\theta}\right) \geq \tilde{a} + S\left(I_H, \tilde{\theta}\right).$$

Type-$\tilde{a}$ firm therefore strictly prefers $(I_H, \tilde{\theta})$ to (I_H, θ):

$$\Pi\left(\tilde{a}, I_H, \tilde{\theta}, P\left(I_H, \tilde{\theta}\right)\right) \geq \tilde{a} + S\left(I_H, \tilde{\theta}\right) > \Pi(\tilde{a}, I_H, \theta, P(I_H, \theta)),$$

leading to a contradiction.

Step 2. For $a < \hat{a}$, $\theta(a)$ satisfies (22).

Since $\theta(a)$ is strictly decreasing, it is sufficient to show $\theta(a)$ has no jump on $[a_{\min}, \hat{a})$, which in turn implies continuity, and Lemma 8 establishes (22).

Suppose in contrast, there exists $a^* < \hat{a}$ such that

$$\bar{\theta} \equiv \lim_{a \uparrow a^*} \theta(a) > \underline{\theta} \equiv \lim_{a \downarrow a^*} \theta(a).$$

For any $a < a^*$,

$$V(a, I_H, \bar{\theta}) < V(a, I_H, \theta(a))$$

and for any $a < a^*$,

$$V(a, I_H, \bar{\theta}) > V(a, I_H, \theta(a)).$$

Lemma 6 implies

$$P(I_H,\bar{\theta}) = a^* + S(I_H,\bar{\theta}).$$

This implies

$$\Pi\left(a^*, I_H, \bar{\theta}, P(I_H,\bar{\theta})\right) = a^* + S(I_H,\bar{\theta}) > a^* + S(I_H,\underline{\theta}).$$

Let $\epsilon < \frac{S(I_H,\bar{\theta})-S(I_H,\underline{\theta})}{2}$ be a small constant. By continuity, there exists a type $a \in (a^*, \hat{a})$ sufficiently close to a^* such that

$$\Pi\left(a, I_H, \bar{\theta}, P(I_H,\bar{\theta})\right) > a^* + S(I_H,\bar{\theta}) - \epsilon,$$

and type-a's equilibrium payoff

$$\Pi^*(a) = a + S(I_H,\theta\left(a\right)) < a^* + S(I_H,\underline{\theta}) + \epsilon.$$

The choice of ϵ implies that type a benefits from deviating to $(\underline{i}, 1)$:

$$\Pi\left(a, I_H, \bar{\theta}, P(I_H,\bar{\theta})\right) > \Pi^*(a),$$

a contradiction.

Step 3. If $\hat{a} > a_{\min}$, then type $a_{\min}$ chooses $\theta = 1$.

Suppose in contrast, type $a_{\min}$ chooses $\theta < 1$. The above implies type $a_{\min}$ is fairly priced and has payoff

$$\Pi^*(a_{\min}) = a_{\min} + S(I_H,\theta).$$

Lemma 6 implies the price associated with the deviation $(I_H, 1)$

$$P(I_H, 1) = a_{\min} + S(I_H, 1),$$

and hence

$$\begin{aligned}\Pi\left(a_{\min}, I_H, 1, P(I_H,1)\right) &= a_{\min} + S\left(I_H, 1\right)\\ &> a_{\min} + S\left(I_H, \theta\right),\\ &= \Pi^*(a_{\min})\end{aligned}$$

that is type $a_{\min}$ benefits from deviating to $(I_H, 1)$, contradiction.

Step 4. $\hat{a}$ is unique.

1. *If $a_{\min}$ and $a_{\max}$ are close enough, such that the ODE (22) and the boundary condition $\check{\theta}\left(a_{\min}\right) = 1$ implies $\check{\theta}(a) > 0$ for all a, then $\hat{a} = a_{\max}$,* that is all types repurchase using methods with efficiency larger than 0.

   Suppose in contrast, $\hat{a} < a_{\max}$. Type $a_{\max}$ prefers $\left(I_H, \check{\theta}\left(a_{\max}\right)\right)$ at price $a_{\max}+S\left(I_H, \check{\theta}\left(a_{\max}\right)\right)$ over $(I_H, 0)$ at the equilibrium price $E\left[a|a \in (\hat{a}, a_{\max}]\right] + S\left(I_H, 0\right)$, since the latter leads to lower surplus and less favorable (meaning lower) market belief:

$$\begin{aligned}
&\Pi\left(a_{\max}, I_H, \check{\theta}\left(a_{\max}\right), a_{\max}+S\left(I_H, \check{\theta}\left(a_{\max}\right)\right)\right) \\
&= a_{\max}+S\left(I_H, \check{\theta}\left(a_{\max}\right)\right) \\
&> a_{\max}+S\left(I_H, 0\right) \\
&= \Pi\left(a_{\max}, I_H, 0, a_{\max}+S\left(I_H, 0\right)\right) \\
&> \Pi\left(a_{\max}, I_H, 0, E\left[a \mid a \in\left(\hat{a}, a_{\max}\right]\right]+S\left(I_H, 0\right)\right)
\end{aligned}.$$

Lemma 6 implies type $\hat{a}$ has the same preference. Moreover, Lemma 8 implies

$$\Pi\left(\hat{a}, I_H, \check{\theta}\left(\hat{a}\right), \hat{a}+S\left(I_H, \check{\theta}\left(\hat{a}\right)\right)\right) > \Pi\left(\hat{a}, I_H, \check{\theta}\left(a_{\max}\right), a_{\max}+S\left(I_H, \check{\theta}\left(a_{\max}\right)\right)\right).$$

Therefore, type $\hat{a}$ strictly prefers $\left(I_H, \check{\theta}\left(\hat{a}\right)\right)$ over $\left(I_H, 0\right)$. By continuity, some type $a \in \left(\hat{a}, a_{\max}\right)$ has the same preference, and hence deviates to $\left(I_H, \check{\theta}\left(\hat{a}\right)\right)$, leading to a contradiction.

2. *If the range of $[a_{\min}, a_{\max}]$ is big enough such that there exists an $a_0 < a_{\max}$ such that $\check{\theta}(a_0) = 0$, then there is a unique $\hat{a} \in [a_{\min}, a_0)$.*

   To see this, notice in this case $\hat{a} \leq a_0$ since $\check{\theta}\left(a\right)$ is not defined for $\hat{a} > a_0$. Moreover,

$$\begin{aligned}
\Pi\left(a_0, I_H, 0, E[a \mid a \in\left(a_0, a_{\max}\right]]+S\left(I_H, 0\right)\right) &> \Pi\left(a_0, I_H, 0, a_0+S\left(I_H, 0\right)\right) \\
&= a_0+S\left(I_H, \check{\theta}(a_0)\right)
\end{aligned}. \tag{C.9}$$

   (a) *If there is $\hat{a} \in (a_{\min}, a_0)$ such that*

$$\Pi\left(\hat{a}, I_H, 0, E[a \mid a \in\left(\hat{a}, a_{\max}\right]]+S\left(I_H, 0\right)\right) = \hat{a}+S\left(I_H, \check{\theta}(\hat{a})\right), \tag{C.10}$$

   then $\hat{a}$ is uniquely determined by (C.10).

   First, there is a unique $\hat{a}$ that satisfies (C.10). Suppose both $\hat{a}_1$ and $\hat{a}_2 < \hat{a}_1$ satisfy (C.10). Lemma 6 implies $\hat{a}_2$ prefers $\left(I_H, \check{\theta}(\hat{a}_2)\right)$ at price $\hat{a}_2+S\left(I_H, \check{\theta}(\hat{a}_2)\right)$ over $\left(I_H, \check{\theta}(\hat{a}_1)\right)$ at price $\hat{a}_1+S\left(I_H, \check{\theta}(\hat{a}_1)\right)$. On the other hand, (C.10) together with Lemma 6 imply type $\hat{a}_2 < \hat{a}_1$ strictly prefer $\left(I_H, \check{\theta}(\hat{a}_1)\right)$ at price $\hat{a}_1+S\left(I_H, \check{\theta}(\hat{a}_1)\right)$ over $\left(I_H, 0\right)$ at price $E[a|a \in (\hat{a}_1, a_{\max}]]+S\left(I_H, 0\right)$, the latter of which it prefers over $\left(I_H, 0\right)$ at price $E[a|a \in (\hat{a}_2, a_{\max}]]+S\left(I_H, 0\right)$. These imply (C.10) holds with $<$ for $\hat{a}_2$.

   Second, $\hat{a} \neq a_{\min}$ since if $\hat{a} = a_{\min}$, the above implies type $a_{\min}$ strictly prefers $\left(I_H, 1\right)$ at price $a_{\min}+S\left(I_H, 1\right)$, which is the equilibrium price according to Lemma 6, over $\left(I_H, 0\right)$ at price $E[a]+S\left(I_H, 0\right)$, which is the equilibrium price. Third, the cutoff type $\hat{a}$ should be indifferent between $\left(I_H, \check{\theta}(\hat{a})\right)$ and $\left(I_H, 0\right)$ (C.10). Otherwise, by continuity of Π, some types around $\hat{a}$ deviate.

   (b) *If there is no $\hat{a} \in (a_{\min}, a_0)$ that satisfies (C.10), then $\hat{a} = a_{\min}$, that is all firms use* $\theta = 0$. This is because otherwise, if $\hat{a} > a_{\min}$, (C.9) implies (C.10) holds with $>$, and some types around $\hat{a}$ who choose $\left(I_H, \check{\theta}(a)\right)$ benefit from deviating to $(I_H, 0)$.

We next prove the existence of the D1 equilibrium under the price as specified in Lemma 6 (i.e., Lemma 3).

Step 1. No firm mimics another type.

It follows Lemma 8 that types with $a > \hat{a}$ do not mimic each other.

If $\hat{a} < a_{\max}$, (9) holds with $\geq$, which implies type $\hat{a}$ weakly prefers $(I_H, 0)$ over $\left(I_H, \check{\theta}(\hat{a})\right)$, and hence over $\left(I_H, \check{\theta}(a)\right)$ for any $a \leq \hat{a}$. Lemma 2 hence implies types $a > \hat{a}$ have the same preference and do not deviate to $\left(I_H, \check{\theta}(a)\right)$ mimicking any $a \leq \hat{a}$.

If $\hat{a} > a_{\min}$, (C.10) holds with $\leq$, which implies type $\hat{a}$ weakly prefers $\left(I_H, \check{\theta}(\hat{a})\right)$ over $(I_H, 0)$. Lemma 4 implies types with $a < \hat{a}$ has the same preference, and therefore do deviate to $(I_H, 0)$ mimicking $a > \hat{a}$.

Step 2. No firm deviates to an off-equilibrium action (i, θ) with $i(1 + \theta b) > I_H(1 + \theta(a_{\min}) b)$ (including $i = 0$).

Lemma 6 implies $P(i, \theta) = a_{\min} + S(i, \theta)$. Meanwhile, $P(I_H, 1) = a_{\min} + S(I_H, 1)$. Therefore, type $a_{\min}$ is fairly priced under both (i, θ) and $(I_H, 1)$. It prefers $(I_H, 1)$ since it leads to higher surplus:

$$\begin{aligned}
\Pi\left(a_{\min}, I_H, 1, P(I_H, 1)\right) &= a_{\min} + S(I_H, 1) \\
&> a_{\min} + S(i, \theta) \\
&= \Pi\left(a_{\min}, i, \theta, P(i, \theta)\right).
\end{aligned}$$

If $\hat{a} > a_{\min}$, $\theta(a_{\min}) = 1$. If $\hat{a} = a_{\min}$, $\theta(a_{\min}) = 0$ or 1, and (C.10) holds with $\geq$, implying $a_{\min}$ weakly prefers $(I_H, \theta(a_{\min}))$ to $(I_H, 1)$. In both cases, type $a_{\min}$ prefers its equilibrium choice $(I_H, \theta(a_{\min}))$ to (i, θ). Lemma 6 then implies all types have the same preference. Since no type mimics type $a_{\min}$ according to Step 1, no type deviates to (i, θ).

Step 3. No firm deviates to an off-equilibrium action (i, θ) with

$$i(1 + \theta b) \in \left[I_H\left(1 + \check{\theta}(\hat{a}) b\right), I_H\left(1 + \theta(a_{\min}) b\right)\right]. \tag{C.11}$$

Condition (C.11) and the continuity of $\check{\theta}(a)$ imply that there is $\bar{a} \in [\hat{a}, a_{\max}]$ such that

$$i(1 + \theta b) = I_H\left(1 + \check{\theta}(\bar{a}) b\right). \tag{C.12}$$

Lemma 6 implies $P(i, \theta) = \bar{a} + S(i, \theta)$. Since $i \geq I_H$, condition (C.12) implies $S(i, \theta) < S\left(I_H, \check{\theta}(\bar{a})\right)$. Type $\bar{a}$ is fairly priced under both (i, θ) and $\left(I_H, \check{\theta}(\bar{a})\right)$, and therefore it weakly prefers $\left(I_H, \check{\theta}(\bar{a})\right)$

over (i,θ). Lemma 6 implies all types weakly prefer $\left(I_H,\check{\theta}(\bar{a})\right)$ over (i,θ). Since no type deviates to $\left(I_H,\check{\theta}(\bar{a})\right)$ according to Step 1, no type deviates to (i,θ).

Step 4. No firm deviates to an off-equilibrium action (i,θ) with $i(1+\theta b)<I_H\left(1+\check{\theta}(\hat{a})b\right)$.

Lemma 6 implies $P(i,\theta)=\hat{a}+S(i,\theta)$. Since $i\geq I_H$, $i\theta b<I_H\check{\theta}(\hat{a})b$. Hence, type $\hat{a}$ is fairly priced under both choices (i,θ) and $\left(I_H,\check{\theta}(\hat{a})\right)$, with the latter generating higher surplus. Therefore, type $\hat{a}$ prefers $\left(I_H,\check{\theta}(\hat{a})\right)$ to (i,θ). Lemma 6 implies types $a<\hat{a}$ have the same preference. Since they do not benefit from mimicking type $\hat{a}$, they do not deviate to (i,θ). If $\hat{a}<a_{\max}$, then (C.10) holds with $\geq$. Type $\hat{a}$ weakly prefers $(I_H,0)$ to $\left(I_H,\check{\theta}(\hat{a})\right)$, and hence to (i,θ). Since $I_H(1+0\cdot b)<i(1+\theta b)$, Lemma 2 implies types $a>\hat{a}$ also prefer $(I_H,0)$ to (i,θ), and hence do not deviate to (i,θ). This completes the proof.

Proof of Proposition 7 for the repurchase game with $\epsilon<\frac{-I_H}{a_{\max}+I_Hb}$:

Notice that the assumptions of footnote 18 imply that *all firms repurchase at least $|I_L|$ in a D1 equilibrium, and as long as firms do not strictly benefit from deviating to $(I_H,1)$, they do not deviate to doing nothing.* This is because firm $a_{\min}$ at most receives $\Pi(a_{\min},0,1,a_{\max})=a_{\max}^{\epsilon}a_{\min}^{1-\epsilon}$ by doing nothing, which is realized when the market attaches price $a_{\max}$ to it. On the other hand, Lemma 7 and that $\epsilon<\frac{-I_H}{a_{\max}+I_Hb}$ imply $\Pi(a,I_H,1,p)$ decreases with p for all p. By choosing $(I_H,1)$, type $a_{\min}$ receives at least $\Pi(a_{\min},I_H,1,a_{\max}+S(I_H,1))$. The assumptions of footnote 18 imply type $a_{\min}$ always prefers $(I_H,1)$ to doing nothing. Since $I_H(1+b)<0$, Lemma 6 implies all firms prefer $(I_H,1)$ to doing nothing.

Step 1. We first show the properties of a D1 equilibrium, and the uniqueness of $\hat{a}$ if $\epsilon\leq\frac{-I_L}{E[a]+I_Lb}$.

1. *All firms choose the most efficient method $\theta=1$.*

   Suppose in a D1 equilibrium outcome, types in a non-empty set A of the firm choose (i,θ) with $\theta<1$. Let $a^*\equiv E[a|a\in A]$. Then

$$\Pi(a^*,i,\theta,P(i,\theta))=P(i,\theta)=a^*+S(i,\theta).$$

   One can find $\left(\tilde{i},\tilde{\theta}\right)$ such that

$$S\left(\tilde{i},\tilde{\theta}\right)>S(i,\theta)$$

   and

$$\Pi\left(a^*,\tilde{i},\tilde{\theta},P\left(\tilde{i},\tilde{\theta}\right)\right)\geq a^*+S\left(\tilde{i},\tilde{\theta}\right)\quad. \tag{C.13}$$

   T*his implies type a^*strictly prefers $\left(\tilde{i},\tilde{\theta}\right)$ to (i,θ):*

$$\begin{aligned}\Pi\left(a^*,\tilde{i},\tilde{\theta},P\left(\tilde{i},\tilde{\theta}\right)\right)&>a^*+S(i,\theta)\\&=\Pi(a^*,i,\theta,P(i,\theta)),\end{aligned}$$

and Lemma 6 then implies some $a \in A$ deviates to $\left(\tilde{i}, \tilde{\theta}\right)$, leading to a contradiction. We show how to find such $\left(\tilde{i}, \tilde{\theta}\right)$ in the following three cases:

(a) Suppose

$$\epsilon\left[a^{*}+S(i, \theta)\right]+i<0.$$

Let $\tilde{\theta} \in(\theta, 1)$ be such that

$$\epsilon\left[a^{*}+S\left(i, \tilde{\theta}\right)\right]+i<0.$$

Then $S\left(i, \tilde{\theta}\right)>S(i, \theta)$. Lemma 7 implies $\Pi\left(a^{*}, i, \tilde{\theta}, p\right)$ decreases in p for $p<a^{*}+S\left(i, \tilde{\theta}\right)$. That $\tilde{\theta}>\theta$ implies $i\left(1+\tilde{\theta} b\right)>i(1+\theta b)$, and Lemma 6 implies

$$P\left(i, \tilde{\theta}\right) \leq a^{*}+S\left(i, \tilde{\theta}\right).$$

This implies (C.13) for $\tilde{i}=i$.

(b) Suppose

$$\epsilon\left[a^{*}+S(i, \theta)\right]+i>0.$$

That $\epsilon<\frac{-I_{H}}{a_{\max }+I_{H} b}$ implies $i>I_{H}$. By continuity, there is $\left(\tilde{i}, \tilde{\theta}\right)$ such that $\tilde{i}\left(1+\tilde{\theta} b\right)<i(1+\theta b)$, $S\left(\tilde{i}, \tilde{\theta}\right)>S(i, \theta)$ and $\epsilon\left[a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)\right]+\tilde{i}>0$. Lemma 7 implies $\Pi\left(a^{*}, \tilde{i}, \tilde{\theta}, p\right)$ increases in p for $p>\epsilon\left[a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)\right]+\tilde{i}$. Lemma 6 implies

$$P\left(\tilde{i}, \tilde{\theta}\right) \geq a^{*}+S\left(\tilde{i}, \tilde{\theta}\right).$$

This implies (C.13).

- Suppose

$$\epsilon\left[a^{*}+S(i, \theta)\right]+i=0.$$

That $\epsilon<\frac{-I_{H}}{a_{\max }+I_{H} b}$ implies $i>I_{H}$. Then there is $\left(\tilde{i}, \tilde{\theta}\right)$ such that $S\left(\tilde{i}, \tilde{\theta}\right)>S(i, \theta)$ and

$$\epsilon\left[a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)\right]+\tilde{i}=0.$$

Lemma 7 implies $\Pi\left(a^{*}, \tilde{i}, \tilde{\theta}, p\right)$ increases with p for $p>a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)$ and decreases with p for $p<a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)$. Therefore,

$$\Pi\left(a^{*}, \tilde{i}, \tilde{\theta}, p\right) \geq \Pi\left(a^{*}, \tilde{i}, \tilde{\theta}, a^{*}+S\left(\tilde{i}, \tilde{\theta}\right)\right)$$

for all p. This implies (C.13).

2. It follows Lemma 6 that $i(a)$ is weakly decreasing, that is, $|i(a)|$ is weakly increasing. There-

fore, there is a cutoff type $\hat{a} \in [a_{\min}, a_{\max}]$ such that firms in the lower interval $\lceil a_{\min}, \hat{a})$ choose $i = I_L$ and firms in the upper interval $(\hat{a}, a_{\max}]$ choose $i < I_L$. One of the two intervals may be empty.

3. *Types in $(\hat{a}, a_{\max}]$ separate on different i.*

   Suppose types in an interval A choose $(i, 1)$ with $i < I_L$. This implies

$$P(i,1) > \inf A + S(i,1) \quad .$$

   If $i > I_H$, Lemma 7 implies

$$\epsilon [\inf A + S(i,1)] + i < \epsilon P(i,1) + i \leq 0. \tag{C.14}$$

   If $i = I_H$, that $\epsilon < \frac{-I_H}{a_{\max}+I_H b}$ implies (C.14).

   By continuity, $\tilde{i}$ in a neighbourhood of i satisfies

$$\epsilon \left[\inf A + S\left(\tilde{i},1\right)\right] + \tilde{i} < 0,$$

   Lemma 6 implies for $\tilde{i} > i$,

$$P\left(\tilde{i},1\right) \leq \inf A + S\left(\tilde{i},1\right).$$

   Lemma 7 implies if type $\inf A$ chooses $\left(\tilde{i},1\right)$, it has payoff

$$\Pi\left(\inf A, \tilde{i}, 1, P\left(\tilde{i},1\right)\right) \geq \inf A + S\left(\tilde{i},1\right).$$

   On the other hand, Lemma 7 and (C.14) imply if type $\inf A$ chooses $(i, 1)$, it has payoff

$$\Pi\left(\inf A, i, 1, P(i,1)\right) < \inf A + S(i,1).$$

   Since

$$\lim_{\tilde{i}\downarrow i} S\left(\tilde{i},1\right) = S(i,1),$$

   there is $\tilde{i} > i$ such that

$$\Pi\left(\inf A, \tilde{i}, 1, P\left(\tilde{i},1\right)\right) > \Pi\left(\inf A, i, 1, P(i,1)\right).$$

   Since Π is continuous in a, there is $a \in A$ such that

$$\Pi\left(a, \tilde{i}, 1, P\left(\tilde{i},1\right)\right) > \Pi\left(a, i, 1, P(i,1)\right),$$

   that is type a deviates to $\left(\tilde{i},1\right)$, leading to a contradiction.

4. *For types $a > \hat{a}$, $i(a)$ satisfies (23).*

Since $i(a)$ is strictly decreasing on $(\hat{a}, a_{\max}]$, it follows step 2 of the proof of Proposition 2[30] that $i(a)$ has no jump. This implies $i(a)$ is continuous on $(\hat{a}, a_{\max}]$, and Lemma 8 establishes (23).

5. *If type $a_{\max}$ chooses $i < I_L$, it chooses $i = I_H$.* The proof follows step 3 of the proof of Proposition 2[30].

6. *$\hat{a}$ satisfies the condition in footnote 20.*

   We show this by cases. Suppose $\hat{a} = a_{\min}$ and $f(a_{\min}) > 0$. Then Lemma 6 implies $P(I_L, 1) = a_{\min} + S(I_L, 1)$, and $f(a_{\min}) > 0$ implies type $a_{\min}$ strictly prefers $(I_L, 1)$ to $\left(\hat{i}(a_{\min}), 1\right)$. Since $\Pi(a, I_L, 1, P(I_L, 1))$ and $\Pi^*(a)$ are both continuous in a, some types $a > a_{\min}$ also prefer $(I_L, 1)$ to $\left(\hat{i}(a), 1\right)$. Hence they deviate to $(I_L, 1)$ leadng to a contradiction.

   Suppose $\hat{a} \in (a_{\min}, a_{\max})$, then that types around $\hat{a}$ do not deviate implies $\hat{a}$ is indifferent between $(I_L, 1)$ and $\left(\hat{i}(a), 1\right)$, hence $f(\hat{a}) = 0$.

   Suppose $\hat{a} = a_{\max}$and $f(a_{\max}) < 0$. Then Lemma 6 implies $P(I_H, 1) = a_{\max} + S(I_H, 1)$, and $f(a_{\max}) < 0$ implies type $a_{\max}$ strictly prefers $(I_H, 1)$ to $(I_L, 1)$. Since $\Pi(a, I_H, 1, P(I_H, 1))$ and $\Pi^*(a)$ are both continuous in a, some types $a > a_{\min}$ also prefer $(I_H, 1)$ to $(I_L, 1)$. Hence they deviate to $(I_H, 1)$ leading to a contradiction.

7. *If $\epsilon \leq \frac{-I_L}{E[a]+I_L b}$ is also satisfied, there is at most one $\hat{a}$ that satisfies the condition in footnote 20.*

   It is sufficient to show that $f(a) = 0$ has at most one solution, $f(a_{\min}) < 0$ implies $f(\tilde{a}) < 0$ for all $\tilde{a}$, and $f(a_{\max}) > 0$ implies $f(\tilde{a}) > 0$ for all $\tilde{a}$.

   That $\epsilon \leq \frac{-I_L}{E[a]+I_L b}$ implies
   $$\epsilon\left[E[a] + S(I_L, 1)\right] + I_L \leq 0.$$

   Lemma 7 implies $\Pi(a, I_L, 1, p)$ decreases in p for $p \leq E[a] + S(I_L, 1)$.

   (a) $f(\tilde{a}) = 0$ has at most one solution.

   Suppose in contrast, $f(a_1) = f(a_2) = 0$ for $a_1 < a_2$. Lemma 8 implies type a_2 strictly prefers $\left(\hat{i}(a_2), 1\right)$ at price $a_2 + S\left(\hat{i}(a_2), 1\right)$ to $\left(\hat{i}(a_1), 1\right)$ at price $a_1 + S\left(\hat{i}(a_1), 1\right)$. That $f(a_1) = 0$ and Lemma 6 imply type a_2 strictly prefers $\left(\hat{i}(a_1), 1\right)$ at price $a_1 + S\left(\hat{i}(a_1), 1\right)$ to $(I_L, 1)$ at price $E\left[a|a \in [a_{\min}, a_1)\right] + S(I_L, 1)$. Since $\Pi(a, I_L, 1, p)$ decreases in p for $p \leq E[a] + S(I_L, 1)$, type a_2 strictly prefers $(I_L, 1)$ at price $E\left[a|a \in [a_{\min} a_1)\right] + S(I_L, 1)$ to $(I_L, 1)$ at price $E\left[a|a \in [a_{\min}, a_2)\right] + S(I_L, 1)$. The above chain implies type a_2 strictly prefers $\left(\hat{i}(a_2), 1\right)$ at price $a_2 + S\left(\hat{i}(a_2), 1\right)$ to $(I_L, 1)$ at price $E\left[a|a \in [a_{\min}, a_2)\right] + S(I_L, 1)$, which implies $f(a_2) < 0$, contradiction.

[30] With the reference to Lemma 3 substituted by Lemma 6.

(b) $f(a_{\min}) < 0$ implies $f(\tilde{a}) < 0$ for all $\tilde{a}$.

If $f(a_{\min}) < 0$, Lemma 6 implies type $\tilde{a} > a_{\min}$ also prefers $(\hat{i}(a_{\min}), 1)$ at price $a_{\min} + S(\hat{i}(a_{\min}), 1)$ to $(I_L, 1)$ at price $a_{\min} + S(I_L, 1)$. Lemma 8 implies type $\tilde{a}$ prefers $(\hat{i}(\tilde{a}), 1)$ at price $\tilde{a} + S(\hat{i}(\tilde{a}), 1)$ to $(\hat{i}(a_{\min}), 1)$ at price $a_{\min} + S(\hat{i}(a_{\min}), 1)$. Since $\Pi(\tilde{a}, I_L, 1, p)$ decreases in p for $p \leq E[a] + S(I_L, 1)$, type $\tilde{a}$ prefers $(I_L, 1)$ at price $a_{\min} + S(I_L, 1)$ to price $E\left[a|a \in [a_{\min}, a)\right] + S(I_L, 1)$. These imply $f(\tilde{a}) < 0$.

(c) $f(a_{\max}) > 0$ implies $f(\tilde{a}) > 0$ for all $\tilde{a}$.

If $f(a_{\max}) > 0$, Lemma 6 implies type $a_{\max}$ strictly prefers $(I_L, 1)$ at price $S(E[\tilde{a}], I_L, 1)$ to $(\hat{i}(\tilde{a}), 1)$ at price $\tilde{a} + S(\hat{i}(\tilde{a}), 1)$ for any $\tilde{a} < a_{\max}$. Lemma 6 implies type $\tilde{a}$ has the same preference. Since $\Pi(\tilde{a}, I_L, 1, p)$ decreases in p for $p \leq E[a] + S(I_L, 1)$, type $\tilde{a}$ prefers $(I_L, 1)$ at price $E[a|a \in [a_{\min}, a)] + S(I_L, 1)$ to price $E[a] + S(I_L, 1)$. Therefore, type $\tilde{a}$ strictly prefers $(I_L, 1)$ at price $E[a|a \in [a_{\min}, a)] + S(I_L, 1)$ to $(\hat{i}(\tilde{a}), 1)$ at price $\tilde{a} + S(\hat{i}(\tilde{a}), 1)$, implying $f(\tilde{a}) > 0$.

Step 2. Remaining to prove is that no firm deviates in the conjectured equilibrium under the D1 price as specified in Lemma 6.

The proof of this follows the same steps as the proof of Proposition 3, with the references to Lemma 2 and 3 substituted by Lemma 6, 4 substituted by Lemma 8, and the references to equation 9 substituted by $-f(\hat{a}) = 0$.

Proof of Proposition 7 for the issue game:

Notice that Lemma 7 implies all firms benefit from an increase in issue price.

The uniqueness of the D1 equilibrium follows identical steps to the proof of Proposition 4 and 5, with the reference to Lemma 2 and 3 substituted by Lemma 6, the reference to footnote 5 substituted by footnote 18, the reference to Lemma 4 substituted by Lemma 4, the reference to ODEs (6) and (15) substituted by (23) and (24).

Proof of Corollary 1:

Let $\epsilon > \tilde{\epsilon}$. ODE (23) implies if $i(a;\epsilon) = i(a;\tilde{\epsilon}) > I_L$,

$$\frac{di(a;\epsilon)}{da} < \frac{di(a;\tilde{\epsilon})}{da} < 0.$$

Let $\hat{a}_\epsilon$ denote the cutoff type $\hat{a}$ when preference for price is ϵ. The above implies $i(a;\epsilon) < i(a;\tilde{\epsilon})$ for $a \in (a_{\min}, \min(\hat{a}_\epsilon, \hat{a}_{\tilde{\epsilon}}))$. Moreover, $\hat{a}_\epsilon < \hat{a}_{\tilde{\epsilon}}$. For $a \in [\hat{a}_\epsilon, \hat{a}_{\tilde{\epsilon}})$,

$$i(a;\epsilon) = I_L < i(a;\tilde{\epsilon}).$$

For $a \geq \hat{a}_{\tilde{\epsilon}}$,

$$i(a;\epsilon) = i(a;\tilde{\epsilon}) = I_L.$$

Therefore, $i(a;\epsilon)$ is non-increasing in ϵ.

Since $\hat{a}_{\epsilon} < \hat{a}_{\tilde{\epsilon}}$, for $a \in (\hat{a}_{\epsilon}, \hat{a}_{\tilde{\epsilon}}]$,

$$\theta(a;\epsilon) < \theta(a;\tilde{\epsilon}) = 1.$$

For $a > \hat{a}_{\tilde{\epsilon}}$, ODE (22) implies if

$$\theta(a;\epsilon) = \theta(a;\tilde{\epsilon}) < 1,$$

then

$$\frac{d\theta(a;\epsilon)}{da} < \frac{d\theta(a;\tilde{\epsilon})}{da} < 0.$$

Therefore, $\theta(a;\epsilon) < \theta(a;\tilde{\epsilon})$ for $a > \hat{a}_{\tilde{\epsilon}}$. For $a < \hat{a}_{\epsilon}$,

$$\theta(a;\epsilon) = \theta(a;\tilde{\epsilon}) = 1.$$

Therefore, $\theta(a;\epsilon)$ is non-increasing in ϵ.

D Supplements to Section 6

To be more specific about the context, we outline the following

The project profitability b is distributed between b_L and b_H. In the issue game, $b_H > b_L > 0$. In the repurchase game, $-1 < b_H < b_L < 0$. The firm maximizes existing shareholders' value, which is written analogous to (2) as a function of the firm's type b, the firm's transaction choices (i, θ) and market belief b^E:

$$\ln \Pi\left(b, i, \theta, b^E\right) = \ln V(b, i, \theta) - \ln\left[1 + \frac{i}{V(b^E, i, \theta) - i}\right], \tag{D.1}$$

where

$$V(b, i, \theta) = a(b) + i + i\theta b$$

is the firm value. In the issue game, $a(b) = a$ is the firm's assets in place which is public information. In the repurchase game, as microfounded in footnote 25,

$$a(b) = a_N + |I_H|(1 - |b|).$$

Proof of Proposition 8 for the issue game:

We first show that in a D1 equilibrium, all types choose size I_H and efficiency 1.

We show this by contradiction. Conjecture an equilibrium in which types in B choose (i, θ) with $|i\theta| < |I_H|$. Since

$$\frac{\partial \ln \Pi\left(b, i, \theta, b^E\right)}{\partial b^E} > 0,$$

for each type b, there is a $b^{\bar{E}}(b)$ such that type b strictly prefers deviating to $(I_H, 1)$ over its equilibrium payoff when the average belief about $(I_H, 1)$ is strictly above $b^{\bar{E}}(b)$. Consider $b^{\bar{E}}(\sup B)$. Since

$$\Pi(\sup B, I_H, 1, \sup B) > \Pi(\sup B, i, \theta, \sup B) > \Pi^*(\sup B), \tag{D.2}$$

and the equilibrium choice of types in B implies the equilibrium belief about $(I_H, 1)$ is such that

$$\Pi\left(\sup B, I_H, 1, b^E\right) \leq \Pi^*(\sup B),$$

$b^{\bar{E}}(\sup B) \in [b_L, b_H)$ and

$$\Pi\left(\sup B, I_H, 1, b^{\bar{E}}(\sup B)\right) = \Pi(\sup B, i, \theta, E[b \in B]).$$

Since $\ln V(b, I_H, 1) - \ln V(b, i, \theta)$ increases with b, this implies for types $b < B$,

$$\ln \Pi\left(b, I_H, 1, b^{\bar{E}}(\sup B)\right) < \ln\Pi(b, i, \theta, E[b \in B]) \leq \ln \Pi^*(b),$$

which implies $b^{\bar{E}}(b) > b^{\bar{E}}(\sup B)$. Therefore, D1 implies the choice $(I_H, 1)$ is only associated with types no smaller than $\sup B$:

$$E[b|(I_H, 1)] \geq \sup B.$$

However, this implies type $\sup B$ strictly prefers $(I_H, 1)$ over (i, θ) by (D.2). This implies some types in B has the same preference. It contradicts that these types choose (i, θ) in equilibrium.

We next show that all types choose $(I_H, 1)$ is a D1 equilibrium. In such an equilibrium,

$$E[b|(I_H, 1)] = E[b].$$

Consider a deviation to $(i, \theta) \neq (I_H, 1)$. Again, since

$$\frac{\partial \ln \Pi\left(b, i, \theta, b^E\right)}{\partial b^E} > 0,$$

for each type b, there is a $b^{\bar{E}}(b)$ such that type b prefers deviating to (i, θ) over its equilibrium payoff when the average belief about (i, θ) is above $b^{\bar{E}}(b)$.

If $b^{\bar{E}}(b) = b_H$ for all b, this implies no type has an incentive to deviate to (i, θ) under any belief.

If $b^{\bar{E}}(b^*) < b_H$ for some b^*, then

$$\Pi\left(b^*, i, \theta, b^{\bar{E}}(b^*)\right) = \Pi^*(b^*) = \Pi(b^*, I_H, 1, E[b]).$$

Since $\ln V(b, i, \theta) - \ln V(b, I_H, 1)$ decreases with b, this implies for $\tilde{b} \gtrless b^*$,

$$\ln \Pi\left(\tilde{b}, i, \theta, b^{\bar{E}}(b^*)\right) \lesseqgtr \ln \Pi\left(\tilde{b}, I_H, 1, E[b]\right) = \ln \Pi^*\left(\tilde{b}\right),$$

implying $b^{\bar{E}}\left(\tilde{b}\right) \gtrless b^{\bar{E}}(b^*)$. D1 therefore requires (i, θ) be associated with the lowest type b_L. Under this belief, no type deviates:

$$\Pi(b, i, \theta, b_L) < \Pi(b, I_H, 1, E[b]),$$

since $(I_H, 1)$ leads to higher surplus and higher market belief.

Proof of Proposition 8 for the repurchase game under the microfoundation of footnote 7:

The keys to the proof are the signs of the following items:

$$\frac{\partial \ln \Pi\left(b, i, \theta, b^E\right)}{\partial b^E} < 0, \tag{D.3}$$

implying firms want to signal low b (i.e., high $|b|$) to repurchase at a cheap price;

$$\frac{\partial S(b, i, \theta)}{\partial \theta} > 0, \tag{D.4}$$

$$\frac{\partial S(b, i, \theta)}{\partial i} < 0; \tag{D.5}$$

where $S(b, i, \theta) = i\theta b$ is the transaction surplus, implying a signal must reduce transaction surplus by decreasing efficiency θ or decreasing repurchase size $|i|$; and

$$\frac{\partial^2 \ln V(b, i, \theta)}{\partial \theta \partial b} < 0, \tag{D.6}$$

$$\frac{\partial^2 \ln V(b, i, 1)}{\partial i \partial b} > 0, \tag{D.7}$$

implying a firm with lower b (i.e., higher $|b|$) loses more from decreasing θ; Given $\theta = 1$, a firm with lower b (higher $|b|$) loses more from decreasing $|i|$.

We first show that in a D1 equilibrium, no type chooses $\theta < 1$. We then show in a D1 equilibrium, no type chooses $i > I_H$. We finally show that all types choose $(I_H, 1)$ is indeed a D1 equilibrium.

1. In a D1 equilibrium, no type chooses $\theta < 1$. Conjecture an equilibrium in which types in B choose (i, θ) with $\theta < 1$. Then consider the market belief associated with $(i, 1)$. (D.3) implies

for each type b, there is $\bar{b^E}(b)$ such that type b strictly prefers to deviate to $(i,1)$ when the market belief of $(i,1)$ is strictly below $\bar{b^E}(b)$, that is the believed $|b|$ is strictly above $\left|\bar{b^E}(b)\right|$.

Consider $\bar{b^E}(\inf B)$. If $(i,1)$ is associated with average belief $E[b \in B]$, then type $\inf B$ strictly prefers $(i,1)$ to (i,θ) since

$$\ln V(b,i,1) > \ln V(b,i,\theta).$$

On the other hand, that types in B choose (i,θ) in equilibrium implies type $\inf B$ weakly prefers (i,θ) to $(i,1)$ under the equilibrium belief. Therefore, $\bar{b^E}(\inf B) \in (b_H, b_L]$ and

$$\Pi\left(\inf B, i, 1, \bar{b^E}(\inf B)\right) = \Pi(\inf B, i, \theta, E[b \in B]) = \Pi^*(\inf B).$$

Since (D.6) implies $\ln V(b,i,1) - \ln V(b,i,\theta)$ decreases with b, for types $b > \inf B$,

$$\Pi\left(b, i, 1, \bar{b^E}(\inf B)\right) < \Pi(b, i, \theta, E[b \in B]) = \Pi^*(b).$$

Therefore, $\bar{b^E}(b) < \bar{b^E}(\inf B)$. Therefore, $(i,1)$ cannot be associated with types higher than $\inf B$. However, under this belief, type $\inf B$ strictly prefers $(i,1)$ to (i,θ) since it leads to higher surplus and better market belief (meaning market belief about higher agency problem $|b|$, which leads to lower repurchase price). By continuity, this implies some types in B have the same preference, leading to a contradiction.

2. In a D1 equilibrium, no type chooses $(i,1)$ with $i > I_H$. Repeat the reasoning in step 1, substituting (i,θ) by $(i,1)$ and $(i,1)$ by $(I_H,1)$. The same process leads to a contradiction. Instead of (D.4) and (D.6), we are now utilizing (D.5) and (D.7).

3. Every type of the firm pooling on $(I_H,1)$ is a D1 equilibrium. In such an equilibrium,

$$E[b|(I_H,1)] = E[b].$$

Consider a choice $(i,\theta) \neq (I_H,1)$. (D.3) implies for each type b, there is a $\bar{b^E}(b)$ such that type b strictly prefers deviating to (i,θ) over its equilibrium payoff when the average belief about (i,θ) is strictly below $\bar{b^E}(b)$.

If $\bar{b^E}(b) = b_L$ for all b, this implies no type has an incentive to deviate to (i,θ) under any belief.

If $\bar{b^E}(b^*) > b_L$ for some b^*, then

$$\Pi\left(b^*, i, \theta, \bar{b^E}(b^*)\right) = \Pi(b^*, I_H, 1, E[b]) = \Pi^*(b^*).$$

(D.6) and (D.7) imply $\ln V(b,i,\theta) - \ln V(b, I_H, 1)$ increases with b. This implies for $b \gtreqless b^*$,

$$\Pi\left(b, i, \theta, \bar{b^E}(b^*)\right) \gtreqless \Pi(b, I_H, 1, E[b]) = \Pi^*(b),$$

implying $\bar{b^E}(b) \gtrless \bar{b^E}(b^*)$. D1 therefore requires (i, θ) be associated with the highest type b_L (i.e., the type with the lowest agency problem $|b_L|$). This is the worst belief since it leads to the highest repurchase price. Under this belief, no type deviates to (i, θ), since $(I_H, 1)$ leads to higher surplus and better market belief.

Proof of Proposition 8 for the repurchase game under the microfoundation of Section 4:

In the repurchase game under the microfoundation of Section 4, b and θ are not multiplicatively separable in firm value $V(b, i, \theta)$. Here we show that Proposition 8 still holds for $[|b_L|, |b_L|] \in [0, 0.49]$.

b is determined by the rate of cash burning, β:

$$b(\beta) = -\left[1 - \frac{1}{\beta T}\left(1 - e^{-\beta T}\right)\right].$$

Let the firm privately observe its rate of cash burning $\beta \in \left[0, \bar{\beta}\right]$.

The firm publicly chooses the frequency of repurchases $n = 1, \ldots, \infty$, which affects the transaction efficiency θ. Since the transaction surplus is

$$S(\beta, i, n) = |i| \left(1 - \frac{1}{n} \frac{1 - e^{-\beta T}}{1 - e^{-\beta \frac{T}{n}}}\right)$$

θ is determined jointly by β and n:

$$\theta(n, \beta) = \frac{S(\beta, i, n)}{|i| \, |b(\beta)|}.$$

Then the firm value (20) can be written as

$$V(\beta, i, n) = a_N + (|I_H| - |i|) - |b(\beta)| (|I_H| - \theta(n, \beta) |i|) \tag{D.8}$$

with $|I_H| = \lambda T$. This is identical to (25) except that $\theta(n, \beta)$ is determined not solely by the firm's choice variable n, but also by the hidden firm type β. We next show that Proposition 8 holds for $\frac{1}{n} \in (0, 1]$ and β with $\beta \in \left(0, \frac{1.56}{T}\right]$, which corresponds to $[|b_L|, |b_L|] \in [0, 0.49]$.

The key is to show conditions analogous to (D.3)-(D.7). Once these are established, the rest of the proof follows identical steps to the proof of the proposition under the microfoundation of footnote 7.

Similar to (D.1), the firm's objective is to maximize

$$\ln \Pi\left(\beta, i, n, \beta^E\right) = \ln V(\beta, i, n) - \ln \left[1 + \frac{i}{V(\beta, i, n) - i}\right].$$

The conditions analogous to (D.3)-(D.5) are

$$\frac{\partial \ln \Pi\left(\beta, i, n, \beta^E\right)}{\partial \beta^E} > 0, \tag{D.9}$$

implying firms want to signal high β to repurchase at a cheap price;

$$\frac{\partial S\left(\beta, i, n\right)}{\partial n} > 0, \tag{D.10}$$

and

$$\frac{\partial S\left(\beta, i, n\right)}{\partial i} < 0, \tag{D.11}$$

implying a signal must reduce surplus by decreasing frequency n or decreasing repurchase size $|i|$. (D.9)-(D.11) can be verified by easy algebra.

The condition analogous to (D.7) is

$$\frac{\partial^2 \ln V\left(\beta, i, \infty\right)}{\partial i \partial \beta} < 0, \tag{D.12}$$

inplying given the maximum frequency $n = \infty$, a firm with higher β loses more from decreasing $|i|$. It holds because

$$V\left(\beta, i, \infty\right) = a_N + \left(|I_H| - |i|\right)\left(1 - |b\left(\beta\right)|\right)$$

implies

$$\frac{\partial \ln V\left(\beta, i, \infty\right)}{\partial i} = \frac{-1}{\frac{a_N}{1-|b(\beta)|} + \left(|I_H| - |i|\right)},$$

and $|b\left(\beta\right)|$ increases with β.

The condition analogous to (D.6) is

$$\frac{\partial^2 \ln V\left(\beta, i, n\right)}{\partial n \partial \beta} > 0, \tag{D.13}$$

implying a firm with higher β loses more from decreasing n.

We next show that (D.13) holds for $\beta \in \left[0, \frac{1.56}{T}\right]$. Since

$$\frac{\partial^2 \ln V\left(\beta, i, n\right)}{\partial n \partial \beta} = \frac{\frac{\partial^2 V}{\partial n \partial \beta}}{V} - \frac{\frac{\partial V}{\partial n} \cdot \frac{\partial V}{\partial \beta}}{V^2},$$

and $\frac{\partial V}{\partial n} > 0$ and $\frac{\partial V}{\partial \beta} < 0$,it is sufficient to show

$$\frac{\partial^2 V\left(\beta, i, n\right)}{\partial n \partial \beta} = \frac{\partial^2 S\left(\beta, i, n\right)}{\partial n \partial \beta} > 0. \tag{D.14}$$

1. As β approaches 0 from the right, (D.14) holds:

$$\lim_{\beta \downarrow 0} \frac{\partial^2 S(\beta, i, n)}{\partial n \partial \beta} = \frac{3n-1}{6n^3} T |i| > 0.$$

2. Since the sign of $\frac{\partial^2 S(\beta,i,n)}{\partial n \partial \beta}$ is independent of the parameters I_H, I_L and a_N, we have verified numerically that for all n with $\frac{1}{n} \in (0,1]$ and β with $\beta \in \left(0, \frac{1.56}{T}\right]$, (D.14) holds.

E Proof of Section 7

Proof of Proposition 9:

Proof. The proof is similar to that of Proposition 2 and 3.

We first show the uniqueness of the D1 equilibrium, and the relation between $\hat{a}$ and $\check{a}$. Lemma 2 implies that better firms issue weakly less. By the assumptions of footnote 5, all types issue. Therefore, there is an upper interval of firms $(\check{a}, a_{\max}]$ that issue the minimum size ($i = I_L$) and a lower interval of firms $[a_{\min}, \check{a})$ that issue more ($i > I_L$). One of the two intervals may be empty.

Step 1. Firms with $a < \check{a}$ separate on different i.

Suppose in contrast, there is an equilibrium issue size $i > I_L$ adopted by a non-singleton set of firms A. Consider type $\tilde{a} > E[a|a \in A]$ in A. Following steps similar to the proof of Proposition 2, one can show that type $\tilde{a}$ benefits from deviating to an issue size marginally lower than i.

Step 2. For $a < \check{a}$, $i(a)$ satisfies (6).

Since $i(a)$ is strictly increasing, it is sufficient to show $i(a)$ has no jump on $[a_{\min}, \check{a})$, which in turn implies continuity, and Lemma 4 establishes (6). That $i(a)$ has no jump on $[a_{\min}, \check{a})$ follows steps similar to the proof of Proposition 2: if there is $a^* < \check{a}$ such that

$$\bar{i} \equiv \lim_{a \uparrow a^*} i(a) > \underline{i} \equiv \lim_{a \downarrow a^*} i(a),$$

then there is a type marginally better than a^* that benefits from deviating to $\underline{i}$, leading to a contradiction.

Step 3. If type $a_{\min}$ chooses $i > I_L$, it chooses $i = I_H$.

Suppose in contrast, type $a_{\min}$ chooses $i \in (I_L, I_H)$, then it is fairly priced both in equilibrium and if it deviates to I_H (implied by Lemma 3). It benefits from deviating to I_H, leading to a contradiction.

Step 4. If $a_{\min}$ and $a_{\max}$ are close enough such that $\hat{i}(a) > I_L$ for all a, then all types issue strictly more than I_L, i.e., $\check{a} = \hat{a} = a_{max}$.

This follows steps similar to the proof of Proposition 2. Suppose $\check{a} < a_{\max}$. Then one can show that type $a_{\max}$ prefers issuing $\hat{i}(a_{\max})$ at price $a_{\max} + S\left(\hat{i}(a_{\max}), 1\right)$ over issuing I_L at the equilibrium price (pooled with lower types). Lemma 2 implies type $\check{a}$ has the same preference strictly. Since Lemma 4 implies type $\check{a}$ strictly prefers issuing $\hat{i}(\check{a})$ at price $\check{a} + S\left(\hat{i}(\check{a}), 1\right)$ over issuing $\hat{i}(a_{\max})$ at price $a_{\max} + S\left(\hat{i}(a_{\max}), 1\right)$, it also strictly prefers issuing $\hat{i}(\check{a})$ at price $\check{a} + S\left(\hat{i}(\check{a}), 1\right)$ over issuing I_L at the equilibrium price. By continuity, some type $a > \check{a}$ has the same preference, i.e., it benefits from deviating to issuing $\hat{i}(\check{a})$. This leads to a contradiction.

Step 5. If $\hat{i}(a) = I_L$ for some $a < a_{\max}$ (which is $\hat{a}$ according to Proposition 5), then $\check{a}$ is unique and $\check{a} < \hat{a}$.

This follows steps similar to the proof of Proposition (3). That $\hat{i}(\hat{a}) = I_L$ implies $\check{a} \leq \hat{a}$. If there is $\check{a} > a_{\min}$ that satisfies

$$\check{a} + S\left(\hat{i}(\check{a}), 1\right) = \Pi\left(\check{a}, I_L, 1, E\left[a | a \in (\check{a}, a_{\max}]\right] + S(I_L, 1)\right), \tag{E.1}$$

then $\check{a}$ is uniquely determined by (E.1). If there is no $\check{a}$ that satisfies (E.1), then $\check{a} = a_{\min}$. Since (E.1) holds for inequality $<$ when $\check{a}$ is substituted by $\hat{a}$, $\check{a} \neq \hat{a}$, and hence $\check{a} < \hat{a}$.

Next we show that under the D1 belief specified in Lemma 3 (i.e., issue size I_L is chosen by firms in $[\check{a}, a_{\max}]$, $i \in \left(I_L, \hat{i}(\check{a})\right)$ is chosen by type $\check{a}$, and $\hat{i}(a)$ for $a \leq \check{a}$ is chosen by type a), no type deviates.

Step 1. Firms in $[a_{\min}, \check{a}]$ do not mimic each other.

It follows Lemma 4.

Step 2. Type $\check{a}$ prefers $\hat{i}(\check{a})$ over $i \in \left(I_L, \hat{i}(\check{a})\right)$.

This is because both choices induce fair prices.

Step 3. Type $\check{a}$ does not deviate. If $\check{a} \in (a_{\min}, a_{\max})$, $\check{a}$ is indifferent between I_L and $\hat{i}(\check{a})$.

If $\check{a} = a_{\min}$, by the definition of $\check{a}$, $\check{a}$ prefers I_L over $\hat{i}(\check{a}) = I_H$. Step 2 implies it prefers I_L over all issue sizes.

If $\check{a} \in (a_{\min}, a_{\max})$, by definition it is indifferent between I_L and $\hat{i}(\check{a})$. By step 1 and step 2, it does not deviate to any issue size.

If $\check{a} = a_{\max}$, it does not deviate to I_L since it is already fairly priced in equilibrium. By step 1 and step 2, it does not deviate to any issue size.

Step 4. Types $a > \check{a}$ do not deviate.

Since type $\check{a} < a_{\max}$, type $\check{a}$ prefers I_L over $i > I_L$ by step 3. Lemma 2 implies types $a > \check{a}$ have the same preference.

Step 5. Types $a < \check{a}$ do not deviate.

Since type $\check{a} > a_{\min}$, type $\check{a}$ preferes $\hat{i}(\check{a})$ over lower issue sizes. Lemma 2 implies types $a < \check{a}$ have the same preference. Since type $a < \check{a}$ prefers $\hat{i}(a)$ over $\hat{i}(\check{a})$ by Lemma 4, it does not deviate to $i < \hat{i}(\check{a})$. By step 1, it does not deviate.

Now we show that for types $a \in (\check{a}, \hat{a}]$ the payoff in this issue game (the restricted game) is higher than in the issue game where firms can choose θ from $[0, 1]$ (the unrestricted game).

Type $\check{a}$ is indifferent between the two games if $\check{a} > a_{\min}$, and has higher payoff in the restricted game if $\check{a} = a_{\min}$. Types $a \in (\check{a}, \hat{a}]$ pool on I_L in the restricted game and separate on $\hat{i}(a)$ in the unrestricted game. Let $\Pi^{\mathrm{R}}(a)$ denote the payoff of type $a \in (\check{a}, \hat{a}]$ in the restricted game,

$$\begin{aligned}
\frac{d}{da} \ln \Pi^{\mathrm{R}}(a) &= \frac{d}{da} \ln \Pi\left(a, I_L, 1, E\left[a | a \in (\check{a}, a_{\max}]\right] + S\left(I_L, 1\right)\right) \\
&= \frac{d}{da} \ln V\left(a, I_L, 1\right) \\
&= \frac{1}{V\left(a, I_L, 1\right)}.
\end{aligned}$$

Let $\Pi^{U}(a)$ denote the payoff of type $a \in (\check{a}, \hat{a}]$ in the unrestricted game. By definition of $\hat{i}(\cdot)$, (8),

$$\begin{aligned}
\frac{d}{da} \ln \Pi^{U}(a) &= \frac{d}{da} \ln \Pi\left(a, \hat{i}(a), 1, a + S\left(\hat{i}(a), 1\right)\right) \\
&= \frac{\partial}{\partial a} \ln V\left(a, \hat{i}(\tilde{a}), 1\right) |_{\tilde{a}=a} \\
&= \frac{1}{V\left(a, \hat{i}(a), 1\right)}.
\end{aligned}$$

This implies $\frac{d}{da} \ln \Pi^{\mathrm{R}}(a) > \frac{d}{da} \ln \Pi^{U}(a)$. Combined with that $\Pi^{\mathrm{R}}(\check{a}) \geq \Pi^{U}(\check{a})$, $\Pi^{\mathrm{R}}(a) > \Pi^{U}(a)$ for $a \in (\check{a}, \hat{a}]$. $\square$