

Automation, financial frictions, and the long-run labor share

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PRELIMINARY

Abstract

We study a minimal model of automation in which, in the limit, labor is replaced by capital in all tasks. If tasks are gross complements and automation is sufficiently slow, the labor share remains strictly positive despite an endogenous reduction in hours worked and full automation in the limit. Heterogeneity in the form of financial frictions naturally produces “workers” and “capitalists”. Workers are right to fear automation: sufficiently high automation rates lower workers’ consumption, both in absolute and relative terms. When complementarities across tasks and between consumption and leisure are strong, financial frictions lower the threshold pace of automation and can therefore facilitate the demise of the labor share: in order to compensate for financial frictions, workers save and work more, leading to lower wage and consumption growth.

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1 Introduction

Technological advance has allowed the automation of many tasks historically performed by workers. Recent progress in artificial intelligence raises the prospect that it may ultimately be possible to automate *all* tasks.

In this paper, we take seriously the possibility that all tasks will ultimately be automated, and analyze the consequences of such automation. The starting point for our analysis is that the consequences of automation depend on evolution of the wages and the returns to capital. A central force in our analysis is Baumol and Bowen (1965)’s insight that productivity improvements in the production of a subset of goods and services increase the price of distinct but complementary goods and services.¹

Specifically, in this paper we analyze a minimal model in which all tasks are asymptotically automated; economic agents are heterogeneous, with “workers” and “capitalists,” endogenously emerging; and in which capital accumulation, labor supply, and factor returns are all determined by standard economic forces. We use this framework to answer a variety of questions, including the following. Do the productivity improvements associated with automation raise or lower labor income? Does automation lead to inequality? If automation leads to a decrease in labor income, to what extent can workers protect themselves by saving and accumulating capital? Do financial frictions that hamper workers’ ability to accumulate capital make the demise of the labor share more or less likely? Does the tendency of workers to work less as they grow richer hinder or promote a fall in the labor share?

Our analysis yields a threshold pace of automation that determines whether or not the labor share of the economy shrinks to zero. For automation speeds below this threshold, Baumol’s effect dominates. That is: Even though all tasks are asymptotically automated, the price of non-automated tasks, and hence wages,

¹Nordhaus (2021) finds “a clear indication of inelasticity of substitution” in US consumption expenditure data. For our results, the key elasticity is between goods—or tasks—produced using labor versus those produced using capital. A large empirical literature estimates the elasticity of substitution between capital around 0.7 and more broadly between 0.5 and 1 (Oberfeld and Raval, 2021; Acemoglu and Restrepo, 2018a).

grows sufficiently fast that the labor share converges to a strictly positive limit.² Indeed, wage growth is sufficiently strong that it *also* outweighs the fact that workers respond to wage growth by reducing their supply of labor. Conversely, automation speeds above the threshold dominate the Baumol effect, so that even though wages grow without bound, the labor share converges to zero. We label the former case as a *stable labor share* equilibrium, and the latter case as a *capital-dominant* equilibrium.

The comparative statics of the speed of automation depend on which side of the aforementioned threshold the economy is. For automation speeds below the threshold, a small increase in the rate of automation increases long-run consumption growth for both workers and capitalists. In contrast, for automation speeds above the threshold, so that capital dominance obtains, further increases in automation lower workers' consumption growth (though increase capitalists' consumption growth). Moreover, for some parameter values capital-dominant and stable labor share equilibria coexist, and comparing consumption growth rates across equilibria yields the same conclusion that workers are worse off in the capital-dominant equilibrium. As such, our analysis suggests that workers are right to fear a fast pace of automation—though should welcome moderate levels of automation.

Faster automation unambiguously decreases both the labor share of the economy, and also the consumption share of workers. As such, faster automation increases inequality, consistent with the fears of many observers.

Workers and capitalists emerge as distinct groups in our analysis solely because they differ in their investment abilities. That is, workers are simply agents who are worse at investing capital. Heterogeneity of investment abilities across individuals is an assumption that enjoys considerable empirical support (e.g., Fagereng et al., 2020; Bach et al., 2020; Smith et al., 2022). The spread in returns across agents reflects financial frictions. In order to keep our analysis as close to a benchmark as possible, we assume that “workers” and “capitalists” are *ex ante* identical in all other respects—in particular, they have identical preferences,³ and identical labor

²This initial implication is inherited from Aghion et al. (2019), on whose specification of the production function our analysis builds.

³Our economy is isomorphic to one in which all agents have the same investment ability, but instead differ in their rate of time preference; under this isomorphism, workers

productivities.

The most striking effect of financial frictions on equilibrium outcomes emerges when both the complementarities of different tasks and of consumption and leisure are both strong. In this case, an increase in financial frictions makes capital dominance more likely. Roughly speaking, greater financial frictions push workers to save more to offset these frictions, and to reduce both consumption and leisure (the two are complements). The associated increase in labor depresses wages, pushing towards capital dominance. Consequently, government policies to reduce financial frictions help workers both directly, and via the equilibrium effect of whether capital dominance occurs.

Our analysis suggests that workers respond to automation by reducing the amount that they work. Although one may be tempted to conclude that this reduction in labor supply contributes to capital dominance, the reverse is in fact the case. To show this, we also consider a perturbation of our model in which labor supply of workers is exogenous and constant; in this perturbation, wages grow slower, and capital dominance occurs for a wider range of parameter values.

Our aim in this paper is to take seriously the prospect that all tasks will eventually be automated, and to analyze the consequences for the economy. As such, we take the speed of automation as exogenous; the key endogenous objects are workers' and capitalists' consumption and labor responses, and the associated equilibrium wages and capital return rates. We conjecture that many of our conclusions would continue to hold with endogenous automation.

A number of the results above depend on consumption and leisure being (gross) complements. The opposite assumption of (gross) substitutes implies that for many parameter values labor increases over time, a prediction at odds with both time-series and cross-country evidence (e.g., Becker, 1965; Huberman and Minns, 2007; Feenstra et al., 2015).⁴ Bick et al. (2018) further find that for most countries, the amount of hours worked is decreasing in the wage.

are simply more impatient agents. While our model does not feature risk, we conjecture that similar results obtain in the presence of risk if agents instead differ in their absolute risk aversion, background risk, or ability to diversify wealth.

⁴Keynes (1930) famously predicted a 15-hour workweek for his grandchildren thanks to rising productivity. In the words of Boppart and Krusell (2020): "As it turned out, Keynes was wildly off quantitatively, but he was right qualitatively (on this issue)."

Related literature:

Our conceptualization of the automation process directly follows the insightful work of Aghion et al. (2019). Relative to that paper, we endogenize both savings and labor supply, and depart from the representative agent framework by introducing financial frictions which in turn generate distinct populations of workers and capitalists. All these features are necessary to address the questions we laid out above.

We follow in the footsteps of earlier models of automation that view technological progress as a gradual replacement of labor with capital as a production factor, such as Zeira (1998) or Acemoglu and Restrepo (2018a,b). In this view, automation extends far beyond artificial intelligence to major sources of economic growth since the Industrial Revolution. From the single-country perspective, automation could also refer to the outsourcing of labor through foreign direct investment in the context of globalization.

Acemoglu and Restrepo (2018b) endogenize automation as well as the invention of new tasks in which labor has a comparative advantage. They find that long-run factor shares are stable if the long-run rental rate of capital is sufficiently high. This result is driven by automation reducing the cost of labor and thereby discouraging further automation and encouraging the development of new tasks. In our case, the labor share stabilizes despite exogenous automation of all tasks in the limit. Our mechanism works through complementarity between tasks (the Baumol effect) and between consumption and leisure. Relative to Acemoglu and Restrepo (2018b), our preference formulation features an explicit bound on leisure and therefore labor supply. Despite these preferences, complementarity interacted with automation can generate a stable labor share consistent with the ‘stylized fact’ of Kaldor (1957).

Another related paper that focuses on heterogeneity from asset returns rather than labor skill is Moll et al. (2022). In their model, automation increases inequality via returns to wealth and by facilitating stagnant wages. As in their model, returns to capital rise with the speed of automation (up to the point of capital dominance), and capital income tends to generate inequality in consumption growth and consumption shares. Unlike in their model, where inequality is driven by stochastic

capital accumulation, our result is obtained from financial frictions that generate the empirically observed differences in returns to capital. We also find that automation *can* reduce wages, but only does so under very specific circumstances. With a low-to-medium speed of automation, wage growth increases with the rate at which labor is replaced by capital in task production.

We further contribute to the long literature on the decline of the labor share following Karabarbounis and Neiman (2014).

Our formulation of the capital-labor complementarity is distinct from the literature explaining changes in skill premia through skill-biased technological change, that is, capital-skill complementarity in production (e.g., Acemoglu, 1998; Krusell et al., 2000; Autor et al., 2003). Instead, we do not take a stance on the types of tasks that remain unautomated for longer, meaning the wages from those tasks may be earned by nurses, teachers, athletes, or—as Baumol and Bowen (1965) would have it—performing artists.

Because the rate of automation in our analysis is exogenously constant, the growth rate of the economy converges in the long-run. In this sense, our analysis doesn’t directly generate a “singularity” in which the growth rate accelerates over time (see Nordhaus (2021), and references therein). But our analysis does suggest that if, for whatever reason, the rate of automation climbs sufficiently high then it overwhelms the economic forces stemming from the complementarity of different tasks, and the complementarity of consumption and leisure, and the labor share of the economy converges to zero.

2 Model

2.1 Preferences and endowments

There is a unit mass of infinitely lived economic agents, each of who continuously consumes, works, and adjusts capital holdings. Each agent discounts the future at rate ρ . There is no uncertainty.

Agents are either “workers” or “capitalists,” with respective measures λ_w and

λ_o .⁵ The only difference between the two groups is that capitalists are weakly more effective at holding capital. Specifically, let $K_{i,t}$ be the date t capital holding of an agent of type $i = w, o$; $L_{i,t}$ the date t time spent working; and $C_{i,t}$ the date t consumption. Moreover, let W_t denote the date t wage rate, and R_t the date t return on capital (not including depreciation and other holding costs). Then capital accumulation for an agent of type t is given by

$$\dot{K}_{i,t} = R_t K_{i,t} + W_t L_{i,t} - \delta_i K_{i,t} - C_{i,t},$$

where δ_i is the combined depreciation and holding costs experienced by type i agents. We assume throughout that

$$\delta_w \geq \delta_o.$$

We normalize an agent's flow endowment of time to 1, so that flow leisure is $1 - L_{i,t}$. Regardless of type, each agent's flow utility is given by

$$\frac{1}{1 - \gamma} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}.$$

Here, ω is a parameter determining the relative importance that agents attach to leisure versus consumption; η is the elasticity of substitution between consumption and leisure; while γ is the standard coefficient from power utility functions. In the special case of $\omega = 0$, the intertemporal elasticity of substitution is $\frac{1}{\gamma}$.

Agents are credit constrained, in the sense that capital holdings cannot be too negative. For simplicity, we set the credit constraint at 0, i.e., $K_{i,t}$ must satisfy

$$K_{i,t} \geq 0.$$

We also impose the standard transversality condition

$$\lim_{t \rightarrow \infty} K_{i,t} \int_0^t e^{-(R_s - \delta)} ds = 0.$$

⁵To avoid confusion with notation for consumption, we subscript quantities relating to capitalists with 'o' (for 'capital owners').

2.2 Technology

Following Aghion et al. (2019), we conceptualize the economy as being composed of a unit measure of complementary “tasks,” with the elasticity of substitution across any pair of tasks equal to σ . For interpretation, a “task” should be interpreted very generally. In contrast to Acemoglu and Restrepo (2018), we think of tasks as being fundamental “needs” such as food, shelter, entertainment, transport etc., so that the set of tasks is unchanging over time.

Importantly, we assume throughout that tasks are gross complements, i.e., $\sigma < 1$. It is this assumption that allows the Baumol force to potentially operate.

At date t , let α_t be the fraction of tasks that has been automated. An automated task is executed using only capital. The remaining fraction $1 - \alpha_t$ of non-automated tasks are executed using only labor.

Let K_t and L_t respectively denote aggregate capital and labor. i.e.,

$$\begin{aligned} K_t &= \lambda_w K_{w,t} + \lambda_o K_{o,t} \\ L_t &= \lambda_w L_{w,t} + \lambda_o L_{o,t}. \end{aligned}$$

Hence date t output is

$$\begin{aligned} F_t = F(K_t, L_t; \alpha_t) &= \left(\alpha_t \left(A_K \frac{K_t}{\alpha_t} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_t) \left(A_L \frac{L_t}{1 - \alpha_t} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\alpha_t^{\frac{1}{\sigma}} (A_K K_t)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_t)^{\frac{1}{\sigma}} (A_L L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \end{aligned} \quad (1)$$

That is: each of the α_t automated tasks receives capital $\frac{K_t}{\alpha_t}$, and each of the $1 - \alpha_t$ non-automated tasks receives labor $\frac{L_t}{1 - \alpha_t}$. Here, A_K and A_L are parameters determining the productivity of capital and labor.

The above expression for output implicitly assumes that once a task can be automated, it is optimal to do so. This is indeed the case provided that aggregate capital K_t is sufficiently large relative to aggregate labor L_t . Since aggregate labor is bounded above by 1, while capital will grow without bound in equilibrium, this condition will always be satisfied after enough time has passed.

For use throughout, note that the marginal products of capital and labor are

$$F_{K,t} = \frac{\partial}{\partial K_t} F(K_t, L_t; \alpha_t) = \alpha_t^{\frac{1}{\sigma}} A_K^{\frac{\sigma-1}{\sigma}} K_t^{-\frac{1}{\sigma}} F_t^{\frac{1}{\sigma}} \quad (2)$$

$$F_{L,t} = \frac{\partial}{\partial L_t} F(K_t, L_t; \alpha_t) = (1 - \alpha_t)^{\frac{1}{\sigma}} A_K^{\frac{\sigma-1}{\sigma}} L_t^{-\frac{1}{\sigma}} F_t^{\frac{1}{\sigma}}. \quad (3)$$

2.3 Automation

As time passes, more and more tasks are automated. Our focus in this paper is on the consequence of automation, so we take the automation process as exogenous. Specifically, we assume a rate of automation of $\theta > 0$, so that

$$\dot{\alpha}_t = (1 - \alpha_t) \theta.$$

Hence asymptotically all tasks are automated,

$$\lim_{t \rightarrow \infty} \alpha_t = 1;$$

but at any finite time t , some tasks remain non-automated.

2.4 Equilibrium

As standard, an equilibrium consist of paths for $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ for $i = w, o$ and rental rates and wages $\{R_t, W_t\}$ such that $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ is individually optimal for each agent taking the path of $\{R_t, W_t\}$ as given, while at each date t rental rates and wages are determined by the competitive conditions

$$\begin{aligned} R_t &= F_K(K_t, L_t; \alpha_t) \\ W_t &= F_L(K_t, L_t; \alpha_t). \end{aligned}$$

2.5 Parameter assumptions

As noted, in order to capture Baumol effects we assume that tasks are gross complements,

$$\sigma < 1.$$

We further assume that consumption and leisure are gross complements,

$$\eta < 1.$$

We have analyzed the economy under the alternative assumption that consumption and leisure are gross substitutes. In this alternative case, many parameter configurations deliver equilibria in which leisure converges to 0, which is inconsistent with observed trends.

Throughout, we assume that for $i = o, w$

$$A_K - \delta_i > \rho > (1 - \gamma)(A_K - \delta_i). \quad (4)$$

The first inequality ensures capital growth, in a benchmark economy with production A_K , while the second inequality ensures the transversality condition is satisfied in the same benchmark.

3 The dynamics of the labor share and capital dominance

We first formally define our notions of a stable labor share and what we will call capital dominance, and note some important implications of these definitions that we use repeatedly in the analysis. We also derive laws of motion for key aggregate quantities: wages F_L , capital return F_K , and output F . This section uses only the definition of the production function (1), and is independent of households' consumption, labor, and capital accumulation choices.

At any time t , the share of output paid to labor is

$$\frac{K_t F_{K,t}}{F_t} = 1 - \frac{L_t F_{L,t}}{F_t}.$$

Among other topics, our analysis sheds light on whether or not the labor share of output remains positive even as all tasks are asymptotically automated. For use throughout:

Definition 1 We say that capital dominance occurs if $\lim_{t \rightarrow \infty} \frac{L_t F_{L,t}}{F_t} = 0$. If instead $\lim_{t \rightarrow \infty} \frac{L_t F_{L,t}}{F_t} > 0$, we say that the labor share is stable.

For the remainder of the paper, we write \lim for $\lim_{t \rightarrow \infty}$.

From (2) and (3), the laws of motion of the rental rate R_t and the wage F_L are

$$\frac{\dot{F}_{K,t}}{F_{K,t}} = \frac{1}{\sigma} \left(\frac{\dot{F}_t}{F_t} - \frac{\dot{K}_t}{K_t} + \theta \frac{1 - \alpha_t}{\alpha_t} \right), \quad (5)$$

$$\frac{\dot{F}_{L,t}}{F_{L,t}} = \frac{1}{\sigma} \left(\frac{\dot{F}_t}{F_t} - \frac{\dot{L}_t}{L_t} - \theta \right), \quad (6)$$

and hence the laws of motion for the capital and labor shares are

$$\frac{\partial}{\partial t} \ln \frac{K_t F_{K,t}}{F_t} = \frac{\dot{K}_t}{K_t} + \frac{\dot{F}_{K,t}}{F_{K,t}} - \frac{\dot{F}_t}{F_t} = (1 - \sigma) \frac{\dot{F}_{K,t}}{F_{K,t}} + \theta \frac{1 - \alpha_t}{\alpha_t}, \quad (7)$$

$$\frac{\partial}{\partial t} \ln \frac{L_t F_{L,t}}{F_t} = \frac{\dot{L}_t}{L_t} + \frac{\dot{F}_{L,t}}{F_{L,t}} - \frac{\dot{F}_t}{F_t} = (1 - \sigma) \frac{\dot{F}_{L,t}}{F_{L,t}} - \theta. \quad (8)$$

Hence capital dominance occurs if, asymptotically,

$$\lim \frac{\dot{F}_{L,t}}{F_{L,t}} < \frac{\theta}{1 - \sigma},$$

while a stable labor share requires

$$\lim \frac{\dot{F}_{L,t}}{F_{L,t}} = \frac{\theta}{1 - \sigma}. \quad (9)$$

Finally, the law of motion of output is given by:

Lemma 1 The law of motion for output is

$$\frac{\dot{F}_t}{F_t} = \frac{K_t F_{K,t}}{F_t} \frac{\dot{K}_t}{K_t} + \frac{L_t F_{L,t}}{F_t} \frac{\dot{L}_t}{L_t} + \frac{\theta}{1 - \sigma} \left(1 - \frac{1}{\alpha_t} \frac{K_t F_{K,t}}{F_t} \right). \quad (10)$$

Substituting Lemma 1 into the law of motion for the wage F_L gives

$$\frac{\dot{F}_{L,t}}{F_{L,t}} = \frac{1}{\sigma} \left(\frac{K_t F_{K,t}}{F_t} \left(\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} \right) + \frac{\theta}{1 - \sigma} \left(\sigma - \frac{1}{\alpha_t} \frac{K_t F_{K,t}}{F_t} \right) \right). \quad (11)$$

Analogously, the law of motion for the rental rate is

$$\frac{\dot{F}_{K,t}}{F_{K,t}} = \frac{1}{\sigma} \left(\frac{L_t F_{L,t}}{F_t} \left(\frac{\dot{L}_t}{L_t} - \frac{\dot{K}_t}{K_t} \right) + \frac{\theta}{1-\sigma} \left(1 - \frac{1}{\alpha_t} \frac{K_t F_{K,t}}{F_t} \right) + \theta \frac{1-\alpha_t}{\alpha_t} \right). \quad (12)$$

We characterize the asymptotic behavior of the economy as the fraction of automated tasks α_t approaches 100%. Throughout, we focus on equilibria in which the capital share has a well-defined and strictly positive limit. When characterizing asymptotic behavior we typically omit all time subscripts.

From the laws of motion for the capital share (7) and the rental rate (5), the rental rate F_K asymptotically converges; define

$$\bar{F}_K \equiv \lim F_K. \quad (13)$$

The capital share is straightforwardly a function of the rental rate:

$$\frac{K_t F_{K,t}}{F_t} = \alpha_t \left(\frac{F_{K,t}}{A_K} \right)^{1-\sigma},$$

and so in particular the limiting capital share is

$$\lim \frac{K F_K}{F} = \left(\frac{\bar{F}_K}{A_K} \right)^{1-\sigma}. \quad (14)$$

Convergence of F_K requires that F and K asymptotically grow at the same rate.

Moreover, since F and K asymptotically grow at the same rate, the law of motion for output, (10), and $\alpha_t \rightarrow 1$ imply that a stable labor share equilibrium exists only if

$$\lim \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) = \frac{\theta}{1-\sigma}. \quad (15)$$

In a capital-dominant equilibrium,

$$\lim \frac{F}{K} = \bar{F}_K = A_K. \quad (16)$$

Finally, and for use throughout: The bounded nature of both labor and leisure

means that, in any equilibrium in which labor L_i has a well-defined asymptotic value, the asymptotic growth rate of leisure is weakly positive and the asymptotic growth rate of labor is weakly negative:

$$\lim \frac{\dot{L}_i}{1 - L_i} \geq 0 \quad (17)$$

$$\lim \frac{\dot{L}_i}{L_i} \leq 0. \quad (18)$$

Moreover, at least one of (17) and (18) must hold with equality.

4 Representative agent benchmark

Both as a benchmark, and to illustrate some key forces, we first analyze a representative agent version of the economy in which workers and capitalists coincide, $\delta_w = \delta_o$. Here, we omit all agent-specific subscripts, and simply write δ for the common depreciation rate.

4.1 Agent optimality and equilibrium conditions

The marginal utilities of consumption and leisure are, respectively,

$$MU_{C,t} = C_t^{-\frac{1}{\eta}} \left(C_t^{\frac{\eta-1}{\eta}} + \omega (1 - L_t)^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\eta\gamma}{\eta-1}} \quad (19)$$

$$MU_{1-L,t} = \omega (1 - L_t)^{-\frac{1}{\eta}} \left(C_t^{\frac{\eta-1}{\eta}} + \omega (1 - L_t)^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\eta\gamma}{\eta-1}}. \quad (20)$$

Hence intratemporal optimality at date t implies

$$W_t C_t^{-\frac{1}{\eta}} \leq \omega (1 - L_t)^{-\frac{1}{\eta}}, \quad (21)$$

with equality if $L_t > 0$, i.e., if leisure is strictly below its maximal level of 1.

Similarly, intertemporal optimality implies

$$\frac{\partial}{\partial t} \ln MU_{C,t} \leq -(R_t - \delta - \rho), \quad (22)$$

with equality if $K_t > 0$, i.e., if the lower bound on capital is non-binding.

In the representative agent benchmark, the zero-work and zero-capital corners $L_t = 0$, $K_t = 0$ cannot arise in equilibrium since this would imply infinitely high wages and rental rates. Hence intratemporal optimality (21) holds with equality, and

$$MU_{C,t} = C_t^{-\frac{1}{\eta}} \left(C_t^{\frac{\eta-1}{\eta}} + \omega \left(\omega^\eta W_t^{-\eta} C_t \right)^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\eta\gamma}{\eta-1}} = C_t^{-\gamma} \left(1 + \omega^\eta W_t^{1-\eta} \right)^{\frac{1-\eta\gamma}{\eta-1}}.$$

Similarly, the intertemporal optimality condition (22) holds with equality, and after substituting in the above expression for $MU_{C,t}$, along with the equilibrium values of W_t and R_t , can be written as

$$-\gamma \frac{\dot{C}_t}{C_t} - (1 - \eta\gamma) \frac{\omega^\eta \frac{\dot{F}_{L,t}}{F_{L,t}}}{F_{L,t}^{\eta-1} + \omega^\eta} = -F_{K,t} + \delta + \rho. \quad (23)$$

Intratemporal optimality (21) at equality also gives

$$\frac{\dot{F}_{L,t}}{F_{L,t}} - \frac{1}{\eta} \frac{\dot{C}_t}{C_t} - \frac{1}{\eta} \frac{\dot{L}_t}{1 - L_t} = 0. \quad (24)$$

In the representative agent benchmark, capital evolves according to

$$\frac{\dot{K}_t}{K_t} = F_{K,t} + \frac{L_t F_{L,t}}{K_t} - \delta - \frac{C_t}{K_t} = \frac{F_t}{K_t} - \delta - \frac{C_t}{K_t}. \quad (25)$$

Hence an equilibrium consists of

$$\{C_t, L_t, K_t, F_t, F_{L,t}, F_{K,t}\}$$

satisfying the intertemporal and intratemporal optimality conditions (23) and (24), and the laws of motion for capital, output, wages, and rental rates, (25), (10), (11), (12).

4.2 Balanced growth

Note that C cannot asymptotically grow faster than F . Since F and K asymptotically grow at the same rate, this in turn implies that C cannot asymptotically grow faster than K .

Moreover, C cannot asymptotically grow slower than K . In this case, $\frac{C}{K} \rightarrow 0$, and so

$$\frac{\dot{K}}{K} \rightarrow \frac{F}{K} - \delta \geq F_K - \delta,$$

which violates the transversality condition. Hence F , K , and C must all grow at the same rate asymptotically. Moreover, the law of motion of capital (25) implies that the growth rate of capital itself has a well-defined limit. Accordingly, define g as the common asymptotic growth rate of K, C, F :

$$g \equiv \lim \frac{\dot{K}}{K} = \lim \frac{\dot{C}}{C} = \lim \frac{\dot{F}}{F}.$$

4.3 Capital dominance versus labor share stability

Next, we separately consider equilibria featuring, respectively, a stable labor share and capital dominance, and characterize the conditions required for each to arise. We start with a stable labor share:

Lemma 2 *In the representative agent benchmark, an equilibrium with a stable labor share exists if*

$$\theta < (1 - \sigma)(A_K - \delta - \rho). \quad (26)$$

The asymptotic growth rate of output, capital, and consumption is

$$g = \frac{\eta\theta}{1 - \sigma}. \quad (27)$$

Wages grow faster than g ,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{g}{\eta}, \quad (28)$$

while labor converges towards 0 according to

$$\lim \frac{\dot{L}}{L} = \left(1 - \frac{1}{\eta}\right) g < 0. \quad (29)$$

The labor share converges towards

$$\lim \frac{LF_L}{F} = 1 - \left(\frac{\delta + \rho + \frac{\theta}{1-\sigma}}{A_K}\right)^{1-\sigma} \quad (30)$$

Next, we consider capital dominance:

Lemma 3 *In the representative agent benchmark, an equilibrium with capital dominance exists if*

$$\theta > (1 - \sigma)(A_K - \delta - \rho). \quad (31)$$

The asymptotic growth rate of output, capital, and consumption is

$$g = \eta(A_K - \delta - \rho). \quad (32)$$

Wages grow faster than g ,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{g}{\eta}, \quad (33)$$

while labor converges towards 0 according to

$$\lim \frac{\dot{L}}{L} = \left(1 - \frac{\sigma}{\eta}\right) g - \theta < 0. \quad (34)$$

Lemmas 2 and 3 generate the following insights for the representative agent economy, the first two of which extend to our main case of heterogeneous agents:

First, capital dominance occurs if and only if the rate of automation θ is sufficiently high.

Second, this critical rate of automation is increasing in the strength of the Baumol effect, which moves inversely with σ . That is: The greater the extent of complementarity across tasks (lower σ) across tasks, and in particular across automated and non-automated tasks, the greater the rate of automation θ required for capital dominance to occur.

Third: Although the asymptotic labor share is decreasing in the rate of task automation θ , the asymptotic growth rate of the economy is increasing. As such, households benefit from automation, at least in the long-run.

As a slight aside: Once the rate of automation θ exceeds the critical value for capital dominance, its precise value ceases to have any effect on either the asymptotic growth rates of the economy or on the consumption-to-output ratio.

5 Heterogeneous agents

With all the above preliminaries in hand, we now turn to our main analysis of an economy with heterogeneous agents. Recall that the sole source of heterogeneity is differential costs of holding capital, $\delta_o < \delta_w$; that is capitalists are strictly more effective capital holders than workers.

5.1 Corner solutions and factor segmentation

The intra- and intertemporal optimality conditions for each group coincide with those in the representative agent case, with the common depreciation rate δ replaced with the group specific rate δ_i for $i = o, w$. We highlight that, in contrast to the representative agent case, the no-work and no-capital-holding corners play an important role in the heterogeneous agent economy, as we shortly establish.

Lemma 4 *The asymptotic leisure growth rate of both groups is 0, i.e., $\lim \frac{\dot{L}_i}{1-L_i} = 0$, and the asymptotic growth rate of wages is strictly positive.*

From Lemma 4, it follows straightforwardly that at least one of the no-work and no-capital-holding corners is relevant:

Corollary 1 *At least one group must be either at the no-capital corner or the no-labor corner.*

Corollary 1 highlights that the intertemporal condition for an agent who holds capital but doesn't work is potentially relevant. At the no-work corner $L_i = 0$,

differentiation of (19) yields

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\frac{1}{\eta} \frac{\dot{C}_i}{C_i} + \frac{1 - \eta\gamma}{\eta} \frac{\frac{\dot{C}_i}{C_i} C_i^{\frac{\eta-1}{\eta}}}{C_i^{\frac{\eta-1}{\eta}} + \omega} = -\frac{\dot{C}_i}{C_i} \frac{\gamma C_i^{\frac{\eta-1}{\eta}} + \frac{\omega}{\eta}}{C_i^{\frac{\eta-1}{\eta}} + \omega}, \quad (35)$$

and so substitution into (22) at equality yields the intertemporal condition for agents who hold capital but don't work:

$$-\frac{\dot{C}_i}{C_i} \frac{\gamma C_i^{\frac{\eta-1}{\eta}} + \frac{\omega}{\eta}}{C_i^{\frac{\eta-1}{\eta}} + \omega} = -(R - \delta_i - \rho). \quad (36)$$

More specifically, the heterogeneous-agent setup with differing capital-holding costs generates a no-work corner for capitalists and a no-capital corner for workers.

Lemma 5 *In a capital-dominant equilibrium, capitalists do not work. In a stable labor share equilibrium, workers do not hold capital.*

5.2 Capital-dominant equilibria

From Lemma 5, in a capital-dominant equilibrium capitalists don't work, and hence workers must do so.

One possibility is that both capitalists and workers hold capital:

Lemma 6 *A capital-dominant equilibrium in which workers hold capital exists if*

$$\theta \geq (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o). \quad (37)$$

The growth rate of group i 's consumption satisfies

$$\lim \frac{\dot{C}_i}{C_i} = \eta(A_K - \delta_i - \rho). \quad (38)$$

Wages grow faster than workers' consumption,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{1}{\eta} \lim \frac{\dot{C}_w}{C_w}, \quad (39)$$

while labor converges towards 0 according to

$$\lim \frac{\dot{L}_w}{L_w} = (\eta - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) - \theta. \quad (40)$$

The second possibility is that workers don't hold capital:

Lemma 7 *A capital-dominant equilibrium in which workers don't hold capital exists if either $\sigma + \eta > 1$ and*

$$\theta \in [(1 - \sigma)(A_K - \delta_o - \rho), (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o)]$$

or if $\sigma + \eta < 1$ and

$$\theta \in [(1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o), (1 - \sigma)(A_K - \delta_o - \rho)].$$

Capitalists' consumption growth satisfies (38), while worker's consumption growth satisfies

$$\lim \frac{\dot{C}_w}{C_w} = \eta \frac{\lim \frac{\dot{C}_o}{C_o} - \theta}{\sigma + \eta - 1} < \lim \frac{\dot{C}_o}{C_o}. \quad (41)$$

Wages grow according to (39), while labor converges towards 0 according to

$$\lim \frac{\dot{L}_w}{L_w} = (\eta - 1) \lim \frac{\dot{F}_L}{F_L}, \quad (42)$$

5.3 Stable labor share equilibria

From Lemma 5, in a stable labor share equilibrium workers don't hold capital, so capitalists do. Equilibria with a stable labor share differ by whether or not capitalists work:

Lemma 8 *A stable labor share equilibrium exists if*

$$\theta < (1 - \sigma)(A_K - \delta_o - \rho). \quad (43)$$

The growth rates of capital and capitalists' and workers' consumption satisfy

$$\lim \frac{\dot{K}_o}{K_o} = \lim \frac{\dot{C}_i}{C_i} = \frac{\eta\theta}{1-\sigma}. \quad (44)$$

Wages grow faster than capital and consumption,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{\theta}{1-\sigma} > \frac{\eta\theta}{1-\sigma}, \quad (45)$$

while labor converges towards 0 according to (42). The labor share converges towards

$$\lim \frac{LF_L}{F} = 1 - \left(\frac{\delta_o + \rho + \frac{\theta}{1-\sigma}}{A_K} \right)^{1-\sigma} \quad (46)$$

5.4 Capital dominance versus labor share stability

We now state our main results, namely a characterization of long-run outcomes as a function of underlying parameters. This characterization depends on combined complementarity of tasks (σ) of consumption and leisure (η). We first consider the case of weak complementarities:

Proposition 1 *If complementarities are weak, $\sigma + \eta > 1$, then capital dominance occurs if and only if the rate of automation θ exceeds*

$$(1-\sigma)(A_K - \delta_o - \rho). \quad (47)$$

For θ below this cutoff,

$$\lim \frac{\dot{C}_o}{C_o} = \lim \frac{\dot{C}_w}{C_w} = \frac{\eta\theta}{1-\sigma} \quad (48)$$

$$\lim \frac{\dot{L}_w}{L_w} = \frac{\eta-1}{1-\sigma}\theta \quad (49)$$

while for θ above this cutoff

$$\lim \frac{\dot{C}_o}{C_o} = \eta(A_K - \delta_o - \rho) \quad (50)$$

$$\lim \frac{\dot{C}_w}{C_w} = \max \left\{ \frac{\eta}{\sigma + \eta - 1} (\eta(A_K - \delta_o - \rho) - \theta), \eta(A_K - \delta_w - \rho) \right\} \quad (51)$$

$$\lim \frac{\dot{L}_w}{L_w} = \max \left\{ \frac{\eta - 1}{\sigma + \eta - 1} (\eta(A_K - \delta_o - \rho) - \theta), (\eta - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) - \theta \right\} \quad (52)$$

Next, we consider the case of strong complementarities:

Proposition 2 *If complementarities are strong, $\sigma + \eta < 1$, then:*

If θ is less than

$$(1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) \quad (53)$$

then the labor share is stable, and consumption and labor satisfy (48) and (49).

If θ lies in the interval

$$((1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o), (1 - \sigma)(A_K - \delta_o - \rho)) \quad (54)$$

then are two equilibria, one of which has a stable labor share and has consumption and labor satisfying (48) and (49), and the other of which is capital-dominant and

$$\lim \frac{\dot{C}_o}{C_o} = \eta(A_K - \delta_o - \rho) \quad (55)$$

$$\lim \frac{\dot{C}_w}{C_w} = \eta(A_K - \delta_w - \rho) \quad (56)$$

$$\lim \frac{\dot{L}_w}{L_w} = (\eta - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) - \theta. \quad (57)$$

Workers' consumption growth is strictly lower, workers labor growth is strictly higher, and capitalists' consumption growth is higher in the capital-dominant equilibrium.

Finally, if θ is above the upper bound of (54) then capital is dominant, and consumption and labor satisfy (55), (56), and (57).

We offer a heuristic explanation for the coexistence of stable labor share and

capital-dominant equilibria for some parameter values. Loosely speaking, capital dominance is associated with lower worker consumption.⁶ The key driver of equilibrium multiplicity is that low consumption is self-reinforcing, as follows. Because of the complementarity of consumption and leisure, low consumption is associated with workers supplying a lot of labor; and this association is especially strong when η is low. The large quantity of labor supplied is in turn associated with low wages. Because task complementarity is strong, the net effect of more labor but lower wages is lower labor income—which in turn leads to low worker consumption.⁷

6 Discussion

6.1 Labor share stability and the rate of automation

Whether or not asymptotic automation of all tasks generates a vanishing labor share depends on the rate of automation, and in particular, whether it is above or below a threshold level. (See Propositions 1 and 2.) This conclusion matches both the representative agent benchmark of Section 4, and also the analysis of the same production function with exogenous labor and saving choices of Aghion et al. (2019).

The comparative static of the threshold rate of automation with respect to the elasticity of substitutability of tasks, σ , is intuitive. The key force pushing the economy towards a stable labor share is Baumol’s, which depends on complementarity of (in our setting) automated and non-automated tasks. Absent complementar-

⁶The formal result in Proposition 2 involves growth rates rather than levels. But here we give a heuristic argument.

⁷When complementarities are strong ($\sigma + \eta < 1$) and stable labor share and capital-dominant equilibria coexist, there are in fact two distinct capital-dominant equilibria. The one we focus on in Proposition 2 is the equilibrium of Lemma 6 in which workers hold capital. But by Lemma 7, there is a second capital-dominant equilibrium in which workers don’t hold capital. The reason that we ignore this equilibrium in Proposition 2 is that it is unstable even within each period precisely when complementarities are strong. The argument is the same as the heuristic argument in the main text for the self-reinforcing nature of a consumption drop; but different from the main text, this heuristic argument is precise in this case, since workers don’t hold capital and so are optimizing period-by-period.

ity (i.e., $\sigma \nearrow 1$), even arbitrarily low rates of automation push the labor share to 0 asymptotically. Greater complementarity (i.e., lower σ) pushes the threshold higher.

At first glance, it might seem surprising that the threshold rate of automation is increasing in the productivity of capital A_K and decreasing in impatience ρ . The economic force behind this result is that, other things equal, higher capital growth rates lower the return on capital, and push against capital dominance. So capital dominance only emerges if the rate of automation θ is strong enough to outweigh the effects of capital accumulation. The rate of capital accumulation in a capital-dominant equilibrium is in turn proportional to $A_K - \delta_o - \rho$, leading to the comparative static.

6.2 Workers and capitalists

Starting from household-specific financial frictions, i.e., $\delta_w > \delta_o$, distinct populations of “workers” and “capitalists” emerge. To be clear, there are equilibria in which workers hold capital, and equilibria in which capitalists work. But our labels are justified by the fact that in all equilibria, workers work strictly more capitalists; and the fraction of capital held by workers is asymptotically negligible.

When the automation rate θ is below the threshold for capital dominance, faster automation raises the consumption growth rates of both groups. Moreover, hours worked converges towards 0 faster as automation becomes faster. So in this range, automation is beneficial for both workers and capitalists.

Workers, however, are correct to fear the consequences of automation rates above the threshold level. For weak complementarities (Proposition 1), workers’ consumption growth falls in the rate of automation. For strong complementarities (Proposition 2), there is a range of automation speeds in which stable labor share and capital-dominant equilibrium coexist, with workers’ consumption growth rates strictly lower in the later case.

Automation’s consequences for inequality are starker: increases in the rate of automation always reduce workers’ share of consumption. Indeed, in capital-dominant equilibria workers’ consumption share asymptotically approaches 0.

Capital-dominant equilibria, in which by definition the labor share is asymptot-

ically vanishing, raise the question of how workers obtain income to consume. The answer depends on the strength of complementarities and the speed of automation. When complementarities are weak and the automation rate isn't too high, all of workers' income derives from labor income. Although the labor share of the economy shrinks, labor income nonetheless grows in absolute terms, enabling a strictly positive consumption growth rate.

But if instead either complementarities are strong and/or automation proceeds sufficiently rapidly, workers' consumption is asymptotically entirely funded by capital income.

Corollary 2 *In any equilibrium in which workers hold capital, workers' capital income grows strictly faster than their labor income.*

6.3 Vanishing hours worked

Regardless of whether or not the labor share of the economy asymptotically vanishes, the number of hours worked do. In this, our analysis is consistent with naïve predictions about the effects of automation that neglect the potential countervailing force of complementarity between automated and non-automated tasks.

Although one might be tempted to conclude that the asymptotic vanishing of hours-worked makes it more likely that the labor share shrinks to zero, the reverse is in fact true. To see this, we briefly consider an exogenous labor supply perturbation of our model, in which workers' labor choice is exogenously fixed at some interior level $\bar{L}_w \in (0, 1)$, while capitalists' labor is exogenously set to zero.

Proposition 3 *Relative to exogenous labor supply, the tendency for workers to respond to automation by working less hinders capital dominance. Holding labor supply fixed lowers the capital-dominance threshold for the pace of automation by a factor of $\eta < 1$, that is, the elasticity of substitution between consumption and leisure.*

Complementarity between consumption and leisure leads workers to contract labor supply in response to consumption growth induced by automation. The capital-dominance threshold is proportional to the IES and to the reciprocal of the

elasticity of substitution between consumption and leisure. In the limit, as consumption grows without bound while labor is bounded, the IES tends to η , such that the two cancel out in the endogenous-labor case. Holding labor supply fixed, only the IES remains, meaning the threshold is lowered by a factor of $\eta < 1$.

6.4 Financial frictions

Financial frictions (i.e., $\delta_w - \delta_o$) directly lower workers' consumption growth whenever workers hold capital. On top of this direct effect, our analysis points to an additional effect of financial frictions when complementarities are strong, namely that they can move the economy from a stable labor share equilibrium to a capital-dominant equilibrium in which workers' consumption growth is discretely lower.

A rough economic intuition for this effect is as follows. Greater financial frictions lower workers' return on saving. Consumption grows in absolute terms in all equilibria, for both workers and capitalists, while leisure converges towards 1. Straightforward calculation establishes that these facts imply that the IES converges to η , which is also the elasticity of substitution between consumption and leisure. Because this parameter is below 1 in our analysis (in order to avoid the counterfactual implication of hours-worked growing over time), the lower return on savings pushes workers to consume less and save more to compensate. Because of the complementarity of leisure and consumption ($\eta < 1$, again), this in turn pushes workers to supply more labor, driving down both wages and the labor share (the latter follows since task complementarity is strong), and hence pushes towards capital dominance.

The implication that small changes in financial frictions can have big effects on workers' consumption growth highlights the importance of the efficiency of the financial system, and (depending on interpretation of the origins of $\delta_w - \delta_o$) financial literacy.

6.5 Size of non-automated sectors

We have discussed how automation affects the long-run dynamics of the labor share at length. Our analysis also has implications for the fraction of GDP stemming

from each non-automated task or “sector,” namely

$$\frac{1}{1-\alpha} \frac{LF_L}{F}.$$

Hence the growth rate of each task’s GDP share is

$$\frac{\partial}{\partial t} \ln \left(\frac{LF_L}{F} \right) + \theta.$$

In a stable labor share equilibrium it follows immediately that each individual non-automated sector grows at a faster rate than the overall economy. Moreover, the same is true in capital-dominant equilibria for a wide range of parameter values. Specifically: In the neighborhood of the capital dominance threshold, the growth rate of the labor share is only slightly negative, and so is dominated by the θ term in the above expression.

Consequently, even on the path to a capital-dominant equilibrium we should often expect to see each non-automated sector grow large relative to the economy before it is finally automated.

7 Conclusion

We study a minimal model of automation in which, in the limit, labor is replaced by capital in all tasks. If tasks are gross complements and automation is sufficiently slow, the labor share remains strictly positive despite an endogenous reduction in hours worked and full automation in the limit. Heterogeneity in the form of financial frictions naturally produces “workers” and “capitalists.” Workers are right to fear automation: sufficiently high automation rates lower workers’ consumption, both in absolute and relative terms. When complementarities across tasks and between consumption and leisure are strong, financial frictions lower the threshold pace of automation and can therefore facilitate the demise of the labor share: in order to compensate for financial frictions, workers save and work more, leading to lower wage and consumption growth.

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A Proofs

Proof of Lemma 1: From the decomposition $F_t = K_t F_{K,t} + L_t F_{L,t}$:

$$\dot{F}_t = \dot{K}_t F_{K,t} + K_t \dot{F}_{K,t} + \dot{L}_t F_{L,t} + L_t \dot{F}_{L,t}$$

and hence (using also (6))

$$\begin{aligned} \frac{\dot{F}_t}{F_t} &= \frac{\dot{K}_t}{K_t} \frac{K_t F_{K,t}}{F_t} + \frac{K_t F_{K,t}}{F_t} \frac{\dot{F}_{K,t}}{F_{K,t}} + \frac{\dot{L}_t}{L_t} \frac{L_t F_{L,t}}{F_t} + \frac{L_t F_{L,t}}{F_t} \frac{\dot{F}_{L,t}}{F_{L,t}} \\ &= \left(\frac{\sigma - 1}{\sigma} \right) \left(\frac{K_t F_{K,t}}{F_t} \frac{\dot{K}_t}{K_t} + \frac{L_t F_{L,t}}{F_t} \frac{\dot{L}_t}{L_t} \right) + \frac{1}{\sigma} \frac{\dot{F}_t}{F_t} + \frac{\theta}{\sigma} \left(\frac{K_t F_{K,t}}{F_t} \frac{1 - \alpha_t}{\alpha_t} - \frac{L_t F_{L,t}}{F_t} \right), \end{aligned}$$

i.e.,

$$\frac{\dot{F}_t}{F_t} = \frac{K_t F_{K,t}}{F_t} \frac{\dot{K}_t}{K_t} + \frac{L_t F_{L,t}}{F_t} \frac{\dot{L}_t}{L_t} + \frac{\theta}{\sigma - 1} \left(\frac{K_t F_{K,t}}{F_t} \frac{1 - \alpha_t}{\alpha_t} - \frac{L_t F_{L,t}}{F_t} \right),$$

which yields the result and completes the proof.

Proof of Lemma 2: Recall that a stable labor share arises if wages asymptotically grow according to (9), and asymptotic capital and labor growth are linked via (15). Note that (9), (15) and the condition that F_K converges imply that the laws of motion (10), (11), (12) for the aggregate quantities F, F_L, F_K hold.

In particular, (9) implies that wages grow without bound,

$$F_L \rightarrow \infty. \tag{58}$$

Moreover, the asymptotic growth rate of leisure must be zero,

$$\lim \frac{\dot{L}}{1 - L} = 0, \tag{59}$$

as follows. If instead $\lim \frac{\dot{L}}{1 - L} > 0$ then intratemporal optimality (24) and the complementarity of labor and leisure ($\eta < 1$) implies that $\lim \frac{\dot{F}_L}{F_L} > g$. But $\lim \frac{\dot{L}}{1 - L} > 0$ also implies that $\lim \frac{\dot{L}}{L} = 0$, and hence (9) and (15) imply that $\lim \frac{\dot{F}_L}{F_L} = g$, a contradiction.

So intratemporal optimality (24) implies that wages grow faster than the economy,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{g}{\eta}. \quad (60)$$

Combining with (9), the growth rate is

$$g = \frac{\eta\theta}{1-\sigma}. \quad (61)$$

Substituting into intertemporal optimality (23) gives the growth rate in terms of the return on capital,

$$g = \eta (\bar{F}_K - \delta - \rho), \quad (62)$$

and so the asymptotic return on capital is

$$\bar{F}_K = \delta + \rho + \frac{\theta}{1-\sigma}. \quad (63)$$

The condition for a stable labor share is simply $\bar{F}_K < A_K$, i.e.,

$$\theta < (1-\sigma)(A_K - \delta - \rho). \quad (64)$$

The growth rate of labor is given by (15),

$$\lim \frac{\dot{L}}{L} = \left(1 - \frac{1}{\eta}\right) g < 0. \quad (65)$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital (25),

$$\lim \frac{C}{K} = \frac{\bar{F}_K}{\lim \frac{KF_K}{F}} - \delta - g = \bar{F}_K^\sigma A_K^{1-\sigma} - \delta - \eta (\bar{F}_K - \delta - \rho), \quad (66)$$

which is strictly positive since $\eta < 1$ and $A_K > \bar{F}_K > \delta$, completing the proof.

Proof of Lemma 3: Under capital dominance, (16) holds, which implies both the laws of motion (10), (12) for the aggregate quantities F, F_K . The law of motion

(11) for F_L simplifies to

$$\lim \frac{\dot{F}_L}{F_L} = \frac{1}{\sigma} \left(g - \lim \frac{\dot{L}}{L} - \theta \right). \quad (67)$$

The capital dominance condition is

$$\lim \frac{\dot{F}_L}{F_L} < g - \lim \frac{\dot{L}}{L}. \quad (68)$$

The asymptotic growth rate of leisure must be zero,

$$\lim \frac{\dot{L}}{1-L} = 0, \quad (69)$$

as follows. The intratemporal optimality (24) condition implies $g < \eta \lim \frac{\dot{F}_L}{F_L}$. If $\lim \frac{\dot{L}}{1-L} > 0$ then $\lim \frac{\dot{L}}{L} = 0$, and the complete automation reduces to $\lim \frac{\dot{F}_L}{F_L} < g$. Since $\eta < 1$, these two bounds on g contradict each other.

So intratemporal optimality (24) implies that wages grow faster than the economy,

$$\lim \frac{\dot{F}_L}{F_L} = \frac{g}{\eta}. \quad (70)$$

In particular, (9) implies that wages grow without bound,

$$F_L \rightarrow \infty. \quad (71)$$

Substituting into intertemporal optimality (23) gives the growth rate in terms of the return on capital, which under complete automation is simply A_K :

$$g = \eta (A_K - \delta - \rho). \quad (72)$$

Combining the two expressions above for the growth rate of wages the growth rate of labor

$$\lim \frac{\dot{L}}{L} = \left(1 - \frac{\sigma}{\eta} \right) g - \theta. \quad (73)$$

Note that the complete automation condition and the expression for the growth

rate of wages directly imply that

$$\lim \frac{\dot{L}}{L} < 0.$$

The complete automation condition rewrites as

$$\frac{g}{\eta} < \frac{\sigma}{\eta}g + \theta, \quad (74)$$

i.e.,

$$\theta > (1 - \sigma)(A_K - \delta - \rho). \quad (75)$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital (25),

$$\lim \frac{C}{K} = A_K - \delta - g, \quad (76)$$

which is strictly positive since $\eta < 1$ and $A_K > \delta$, completing the proof.

Proof of Lemma 4: First, there cannot be an equilibrium in which some group i both holds capital and has $\lim \frac{\dot{L}_i}{1-L_i} > 0$, as follows. Suppose to the contrary that such an equilibrium exists. From the law of motion for capital,

$$\lim \frac{\dot{K}_i}{K} = \bar{F}_K - \delta_i + \lim \frac{L_i F_L - C_i}{K_i}.$$

Since group i holds capital, its transversality condition can hold only if the final term on the RHS is non-positive, which in turn requires

$$\lim \frac{\dot{C}_i}{C_i} \geq \lim \frac{\dot{L}_i}{L_i} + \lim \frac{\dot{F}_L}{F_L} = \lim \frac{\dot{F}_L}{F_L},$$

where the equality follows from the supposition that $\lim \frac{\dot{L}_i}{1-L_i} > 0$. But since group i isn't at the no-work corner, the analogue of (24) implies that

$$\lim \frac{\dot{F}_L}{F_L} = \frac{1}{\eta} \left(\lim \frac{\dot{C}_i}{C_i} + \lim \frac{\dot{L}_i}{1-L_i} \right) > \frac{\dot{C}_i}{C_i},$$

where the inequality follows from $\eta < 1$ and the supposition that $\lim \frac{\dot{L}_i}{1-L_i} > 0$.

But these last two inequalities are mutually contradictory, establishing the claim.

Second, there cannot be an equilibrium in which some group i doesn't hold capital and has $\lim \frac{\dot{L}_i}{1-L_i} > 0$, as follows. Suppose to the contrary that such an equilibrium exists. So the intratemporal optimality and budget constraint for this non-capital-holding group i are

$$\begin{aligned} \frac{\dot{C}_i}{C_i} + \frac{\dot{L}_i}{1-L_i} &= \eta \frac{\dot{F}_L}{F_L} \\ \frac{\dot{C}_i}{C_i} &= \frac{\dot{F}_L}{F_L}. \end{aligned}$$

Hence

$$\frac{\dot{L}_i}{1-L_i} = (\eta - 1) \frac{\dot{F}_L}{F_L}.$$

Since $\lim \frac{\dot{L}_i}{1-L_i} \geq 0$ this implies

$$\lim \frac{\dot{C}_i}{C_i} = \lim \frac{\dot{F}_L}{F_L} \leq 0. \quad (77)$$

Regardless of whether $\lim \frac{\dot{F}_L}{F_L} = 0$ or $\lim \frac{\dot{F}_L}{F_L} < 0$, analogously to (23) the condition for group i to not hold capital asymptotically is

$$-\gamma \lim \frac{\dot{C}_i}{C_i} < -(A_K - \delta_i - \rho), \quad (78)$$

i.e.,

$$\lim \frac{\dot{C}_i}{C_i} > \frac{1}{\gamma} (A_K - \delta_i - \rho) > 0,$$

where the second inequality follows from assumption (4). This contradicts (77), establishing the claim.

So far, we have established that $\lim \frac{\dot{L}_i}{1-L_i} = 0$ for both groups. We now show that $\lim \frac{\dot{F}_L}{F_L} > 0$. At least one group i must work, and the intratemporal optimality condition for this group gives

$$\lim \frac{\dot{C}_i}{C_i} = \eta \lim \frac{\dot{F}_L}{F_L}.$$

Suppose to the contrary that $\lim \frac{\dot{F}_L}{F_L} \leq 0$. Then one obtains a contradiction by exactly the same steps as above, completing the proof.

Proof of Corollary 1: Suppose to the contrary that both groups have strictly positive capital holdings and exert strictly positive labor, even for time t arbitrarily large. Then (23) and (24) hold for each group, with δ replaced by its group specific values in (23).

It is immediate that it is impossible to have $\lim \frac{\dot{L}_i}{1-L_i} = 0$ for both groups $i = o, w$.

Exactly the same argument at the start of Lemma 2 implies that there cannot be a stable labor share equilibrium in which $\lim \frac{\dot{L}_i}{1-L_i} > 0$ for some group $i = o, w$.

Lemma 4 establishes that there cannot be an equilibrium in which $\lim \frac{\dot{L}_i}{1-L_i} > 0$ for some group $i = o, w$, completing the proof.

Proof of Lemma 5: For capital dominance: If workers hold capital, then capitalists must be in the no-work corner by Lemma 1. From (21) for capitalists, a corner solution with $L_{o,t} = 0$ requires

$$C_{o,t} > \left(\frac{W}{\omega}\right)^\eta = \frac{C_{w,t}}{1-L_{w,t}} \quad (79)$$

where the second equality follows from intratemporal optimality (21) for workers (since one group must work). If workers do not hold capital and consume hand-to-mouth their consumption relative to aggregate capital tends to zero

$$\frac{C_w \lambda_w}{K} = \frac{F_L L_w \lambda_w}{K} \leq \frac{F_L L F}{F K} = \frac{F_L L}{F} A_K \rightarrow 0 \quad (80)$$

Scaling both sides of (79) by K , the right-hand side becomes $\frac{C_{w,t}}{(1-L_{w,t})K}$, which goes to zero by (80) and Lemma 4, which states that $L_w \rightarrow 0$. By the transversality condition of capitalists, the left-hand side is bounded away from zero, thus concluding the proof for capital-dominant equilibria.

For stable labor share: Suppose both capitalists and workers hold capital, then

by intratemporal optimality of workers

$$\frac{\dot{C}_w}{C_w} = \eta \frac{\dot{F}_L}{F_L}. \quad (81)$$

Combining this with intertemporal optimality for both groups, the limit (as $F_L \rightarrow \infty$) requires

$$\frac{1}{\eta} \frac{\dot{C}_o}{C_o} = F_K - \delta_o - \rho \quad (82)$$

$$\frac{1}{\eta} \frac{\dot{C}_w}{C_w} = F_K - \delta_w - \rho, \quad (83)$$

meaning $\frac{\dot{C}_o}{C_o} > \frac{\dot{C}_w}{C_w}$, since $\delta_o < \delta_w$. To satisfy group-specific transversality, capital growth cannot exceed consumption growth, $\frac{\dot{K}_i}{K_i} \leq \frac{\dot{C}_i}{C_i}$. An equilibrium with $\frac{\dot{K}_o}{K_o} > \frac{\dot{K}_w}{K_w}$ is not possible, because, with a stable labor share, labor income asymptotically grows with aggregate capital, that is, at rate $\frac{\dot{K}_o}{K_o} > \frac{\dot{C}_w}{C_w}$, which would violate the workers' transversality condition. This concludes the proof for stable labor share equilibria.

Proof of Lemma 6: We characterize the conditions for a capital-dominant equilibrium in which both groups hold capital to exist. In a capital-dominant equilibrium, $R \rightarrow A_K$. From (36), and using $C_o \rightarrow \infty$, an owner's intertemporal optimality condition is asymptotically

$$-\frac{1}{\eta} \lim \frac{\dot{C}_o}{C_o} = -(A_K - \delta_o - \rho). \quad (84)$$

Applying Lemma 4, a worker's inter- and intratemporal optimality conditions are asymptotically

$$\begin{aligned} -\gamma \lim \frac{\dot{C}_w}{C_w} - (1 - \eta\gamma) \lim \frac{\dot{F}_L}{F_L} &= -A_K + \delta_w + \rho \\ \lim \frac{\dot{C}_w}{C_w} &= \eta \lim \frac{\dot{F}_L}{F_L}, \end{aligned}$$

the laws of motion for capital are

$$\begin{aligned}\lim \frac{\dot{K}_o}{K_o} &= A_K - \delta_o - \lim \frac{C_o}{K_o} \\ \lim \frac{\dot{K}_w}{K_w} &= A_K - \delta_w + \lim \frac{L_w F_L - C_w}{K_w}\end{aligned}$$

and the law of motion for wages is

$$\lim \frac{\dot{F}_L}{F_L} = \frac{1}{\sigma} \left(\lim \frac{\dot{K}}{K} - \lim \frac{\dot{L}_w}{L_w} - \theta \right).$$

From the worker's optimality conditions,

$$\begin{aligned}\lim \frac{\dot{C}_w}{C_w} &= \eta (A_K - \delta_w - \rho) \\ \lim \frac{\dot{F}_L}{F_L} &= A_K - \delta_w - \rho,\end{aligned}$$

where both the above growth rates are strictly positive by assumption (4).

An owner's transversality condition implies that C_o and K_o must asymptotically grow at the same rate, which from an owner's intertemporal optimality condition is

$$\lim \frac{\dot{C}_o}{C_o} = \lim \frac{\dot{K}_o}{K_o} = \eta (A_K - \delta_o - \rho).$$

We characterize an equilibrium in which C_w and K_w asymptotically grow at the same rate. In this case,

$$\lim \frac{\dot{K}_w}{K_w} < \lim \frac{\dot{K}_o}{K_o} = \lim \frac{\dot{K}}{K},$$

and so

$$\lim \frac{\dot{L}_w}{L_w} = \eta (A_K - \delta_o - \rho) - \sigma (A_K - \delta_w - \rho) - \theta.$$

A worker's transversality condition is equivalent to

$$\lim \frac{\dot{C}_w}{C_w} \geq \lim \frac{\dot{F}_L}{F_L} + \lim \frac{\dot{L}_w}{L_w}, \quad (85)$$

which substituting in the above expressions is equivalent to

$$\eta(A_K - \delta_w - \rho) \geq A_K - \delta_w - \rho + \eta(A_K - \delta_o - \rho) - \sigma(A_K - \delta_w - \rho) - \theta,$$

and hence to

$$\theta \geq (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o). \quad (86)$$

Note that $\lim \frac{\dot{C}_o}{C_o} > \lim \frac{\dot{C}_w}{C_w}$ together with the worker transversality condition (85) implies that the capital-dominance condition is satisfied; and also implies that capitalists indeed don't work. Moreover, the worker transversality condition ensures that $\lim \frac{\dot{L}_w}{L_w} < 0$.

Proof of Lemma 7: We characterize the conditions for a capital-dominant equilibrium in which workers don't hold capital to exist. In a capital-dominant equilibrium, $R \rightarrow A_K$. Parallel to the proof of Lemma 6, the asymptotic equilibrium conditions are

$$\begin{aligned} -\frac{1}{\eta} \lim \frac{\dot{C}_o}{C_o} &= -(A_K - \delta_o - \rho) \\ \lim \frac{\dot{C}_w}{C_w} &= \eta \lim \frac{\dot{F}_L}{F_L} \\ \lim \frac{\dot{K}_o}{K_o} &= A_K - \delta_o - \lim \frac{C_o}{K_o} \\ \lim \frac{\dot{C}_w}{C_w} &= \lim \frac{\dot{F}_L}{F_L} + \lim \frac{\dot{L}_w}{L_w} \\ \lim \frac{\dot{F}_L}{F_L} &= \frac{1}{\sigma} \left(\lim \frac{\dot{K}_o}{K_o} - \lim \frac{\dot{L}_w}{L_w} - \theta \right). \end{aligned}$$

Relative to the proof of Lemma 6, there is no longer an intertemporal optimality equality for workers, and the law of motion for a worker's capital is replaced with the intratemporal budget constraint.

As in the proof of Lemma 6,

$$\lim \frac{\dot{C}_o}{C_o} = \lim \frac{\dot{K}_o}{K_o} = \eta(A_K - \delta_o - \rho).$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim \frac{\dot{L}_w}{L_w} = (\eta - 1) \lim \frac{\dot{F}_L}{F_L}.$$

Hence

$$\lim \frac{\dot{F}_L}{F_L} = \frac{\lim \frac{\dot{K}_o}{K_o} - \theta}{\sigma + \eta - 1}.$$

The capital-dominance condition is $\lim \frac{\dot{K}_o}{K_o} > \lim \frac{\dot{F}_L}{F_L} + \lim \frac{\dot{L}_w}{L_w}$. Note that if the capital-dominance condition holds then $\lim \frac{\dot{C}_o}{C_o} > \lim \frac{\dot{C}_w}{C_w}$, which ensures that capitalists indeed don't work asymptotically. Substituting in, the capital-dominance condition is

$$\lim \frac{\dot{K}_o}{K_o} > \eta \frac{\lim \frac{\dot{K}_o}{K_o} - \theta}{\sigma + \eta - 1}.$$

The condition that workers asymptotically don't want to hold capital is (using (23))

$$-\gamma \lim \frac{\dot{C}_w}{C_w} - (1 - \eta\gamma) \lim \frac{\dot{F}_L}{F_L} \leq -(A_K - \delta_w - \rho),$$

which using workers' intratemporal optimality and then substituting in for the growth of wages simplifies to

$$\lim \frac{\dot{F}_L}{F_L} = \frac{\lim \frac{\dot{K}_o}{K_o} - \theta}{\sigma + \eta - 1} \geq A_K - \delta_w - \rho = \frac{1}{\eta} \lim \frac{\dot{K}_o}{K_o} - (\delta_w - \delta_o).$$

Note that the above condition combined with (4) ensures that $\lim \frac{\dot{L}_w}{L_w} < 0$.

Hence an equilibrium of this type exists if either $\sigma + \eta > 1$ and

$$\theta \in \left[\frac{1 - \sigma}{\eta} \lim \frac{\dot{K}_o}{K_o}, \frac{1 - \sigma}{\eta} \lim \frac{\dot{K}_o}{K_o} + (\sigma + \eta - 1)(\delta_w - \delta_o) \right]$$

or if $\sigma + \eta < 1$ and

$$\left[\frac{1 - \sigma}{\eta} \lim \frac{\dot{K}_o}{K_o} + (\sigma + \eta - 1)(\delta_w - \delta_o), \frac{1 - \sigma}{\eta} \lim \frac{\dot{K}_o}{K_o} \right]$$

Substituting in for $\lim \frac{\dot{K}_o}{K_o}$ yields the result.

Proof of Lemma 8: We characterize the conditions for a stable labor share equilibrium to exist. Parallel to the proof of Lemma 7, but adding a weak inequality for the owner's intratemporal optimality condition, the asymptotic equilibrium conditions are

$$\begin{aligned}
\lim \frac{\dot{C}_o}{C_o} &\geq \eta \lim \frac{\dot{F}_L}{F_L} \\
-\frac{1}{\eta} \lim \frac{\dot{C}_o}{C_o} &= -(\bar{F}_K - \delta_o - \rho) \\
\lim \frac{\dot{C}_w}{C_w} &= \eta \lim \frac{\dot{F}_L}{F_L} \\
\lim \frac{\dot{K}_o}{K_o} &= \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o} \\
\lim \frac{\dot{C}_w}{C_w} &= \lim \frac{\dot{F}_L}{F_L} + \lim \frac{\dot{L}_w}{L_w} \\
\lim \frac{\dot{F}_L}{F_L} &= \frac{\theta}{1 - \sigma}.
\end{aligned}$$

Relative to the proof of Lemma 7, the intratemporal optimality condition of capitalists holds if capitalists work. Further, the law of motion of the wage is replaced by (9) given the requirement for a stable labor share.

From the owner's intertemporal optimality condition,

$$\lim \frac{\dot{C}_o}{C_o} = \lim \frac{\dot{K}_o}{K_o} = \eta (\bar{F}_K - \delta_o - \rho).$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim \frac{\dot{L}_w}{L_w} = (\eta - 1) \lim \frac{\dot{F}_L}{F_L} = (\eta - 1) \frac{\theta}{1 - \sigma}.$$

Note that this condition ensures that $\lim \frac{\dot{L}_w}{L_w} < 0$. Further, from (15), a stable labor share requires

$$\lim \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) = \frac{\theta}{1 - \sigma}.$$

Since workers don't hold capital, $\frac{\dot{K}}{K} = \frac{\dot{K}_o}{K_o}$. Assume for now (proof below) that, if capitalists work at all, their labor growth rate equals that of workers, that is, $\lim \frac{\dot{L}}{L} = \lim \frac{\dot{L}_o}{L_o} = \lim \frac{\dot{L}_w}{L_w}$. Combining,

$$\lim \frac{\dot{K}_o}{K_o} = \eta \frac{\theta}{1 - \sigma} = \eta (\bar{F}_K - \delta_o - \rho),$$

where the first equality follows from (15) and the second from balanced growth and an owner's intertemporal optimality. Note that this establishes the growth rate of capital. The limiting capital rental rate is therefore given by

$$\bar{F}_K = \frac{\theta}{1 - \sigma} + \delta_o + \rho. \quad (87)$$

For the asymptotic capital share to be bounded away from one,

$$\left(\frac{\bar{F}_K}{A_K} \right)^{1 - \sigma} < 1,$$

which, from (87), is equivalent to

$$\frac{\theta}{1 - \sigma} + \delta_o + \rho < A_K.$$

Rearranging establishes the claim for the stable labor share condition, (43).

Now to show that aggregate labor growth is indeed equal to that of workers, note that if capitalists work, $\lim F_L L_o > 0$. Combining the law of motion for capital with intertemporal optimality yields

$$\bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o} = \eta (\bar{F}_K - \delta_o - \rho).$$

Combining with (87) and rearranging gives

$$\lim \frac{C_o - F_L L_o}{K_o} = (1 - \eta) \frac{\theta}{1 - \sigma},$$

Since C_o and K_o grow at the same rate asymptotically (equal to $\frac{\theta}{1 - \sigma}$, any nonzero

labor income earned by capitalists must also grow at that same rate, i.e.,

$$\lim \frac{\dot{L}_o}{L_o} + \lim \frac{\dot{F}_L}{F_L} = \lim \frac{\dot{C}_o}{C_o} = \eta \lim \frac{\dot{F}_L}{F_L} = \lim \frac{\dot{L}_w}{L_w},$$

where the second equality follows from intratemporal optimality of working capitalists, and the third from intratemporal optimality of workers. Note that this also establishes the claims that consumption growth rates are equal across groups and, since $\eta < 1$ below the rate of wage growth.

The expression for the limiting labor share follows from steps analogous to those in the proof of Lemma 2, using the intertemporal optimality condition and capital accumulation equation for capitalists rather than the representative agent.

Proof of Proposition 1: Immediate from prior results.

Proof of Proposition 2: The result is largely immediate from prior results. The comparison of labor growth rates when θ falls in the intermediate range 54 follows from the algebra below:

First, note that since the labor growth rate is a linear function of θ in both the stable labor share and capital-dominant equilibria, it suffices to compare the labor growth rates at the end points of the interval 54.

At the lower end of the interval, $\theta = (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o)$, and in the stable labor equilibrium

$$\lim \frac{\dot{L}_w}{L_w} = \frac{\eta - 1}{1 - \sigma} \theta = (\eta - 1)(A_K - \delta_w - \rho) + \frac{\eta(\eta - 1)}{1 - \sigma} (\delta_w - \delta_o),$$

while in the capital-dominant equilibrium,

$$\begin{aligned} \lim \frac{\dot{L}_w}{L_w} &= (\eta - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) - \theta \\ &= (\eta - 1)(A_K - \delta_w - \rho), \end{aligned}$$

which is strictly greater.

At the upper end of the interval, $\theta = (1 - \sigma)(A_K - \delta_o - \rho)$, and in the stable

labor equilibrium

$$\lim \frac{\dot{L}_w}{L_w} = \frac{\eta - 1}{1 - \sigma} \theta = (\eta - 1) (A_K - \delta_o - \rho),$$

while in the capital-dominant equilibrium,

$$\begin{aligned} \lim \frac{\dot{L}_w}{L_w} &= (\eta - \sigma) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o) - \theta \\ &= (\eta - \sigma) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o) \\ &\quad - (1 - \sigma) (A_K - \delta_w - \rho) - (1 - \sigma) (\delta_w - \delta_o) \\ &= (\eta - 1) (A_K - \delta_w - \rho) + (\sigma + \eta - 1) (\delta_w - \delta_o) \\ &= (\eta - 1) (A_K - \delta_o - \rho) + \sigma (\delta_w - \delta_o), \end{aligned}$$

which again is strictly greater. This completes the proof.

Proof of Corollary 2: The only case in which workers hold capital is characterized in Lemma 6. Workers' labor income grows at rate $\frac{\dot{L}_w}{L_w} + \frac{\dot{F}_L}{F_L}$, which evaluating equals

$$(\eta - \sigma + 1) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o) - \theta + (A_K - \delta_w - \rho).$$

Since this equilibrium requires (37) to hold, workers' labor income grows at rate that is less than

$$\eta (A_K - \delta_w - \rho),$$

which in turn equals the growth rate of worker's consumption. The result follows, completing the proof.

Proof of Proposition 3: We do not remove leisure from the agents' preferences but instead set constant labor supplies $L_o = 0$ and $L_w = L \in (0, 1)$.⁸ Setting labor

⁸In the limit as consumption grows unbounded but leisure is bounded, the IES with consumption and leisure as gross complements tends to η rather than $1/\gamma$. Retaining this feature of the preferences in the exogenous-labor case facilitates comparison with the endogenous-labor case.

supply growth to zero in (11) in the limit of $\alpha_t \rightarrow 1$ and $K_t F_K / F \rightarrow 1$ gives

$$\lim \frac{\dot{F}_L}{F_L} = \frac{1}{\sigma} \left(\lim \frac{\dot{K}}{K} - \theta \right). \quad (88)$$

Intertemporal optimality of agent i , combined with the transversality condition requires

$$\frac{\dot{K}_i}{K_i} = \eta (\bar{F}_K - \delta_i - \rho)$$

Aggregate capital growth is determined by the intertemporal optimality condition of the agent accumulating capital faster, i.e., capitalists. (Whether or not workers hold capital does not change this result: if they consume hand-to-mouth, their intertemporal optimality condition does not hold. If they do, their capital growth is slower, as $\delta_w > \delta_o$.)

As previously, the condition for capital dominance is that labor income asymptotically grows slower than capital income. In this case, labor supply is constant and as before, the return on capital converges to its capital-dominance limit A_K . So,

$$\lim \frac{\dot{F}_L}{F_L} < \frac{\dot{K}}{K}$$

Using capital growth of capitalists in (88), this condition becomes

$$\theta > \tilde{\theta}_{CD} \equiv \eta(1 - \sigma)(A_K - \delta_o - \rho) \quad (89)$$

Denoting the analogous threshold in the endogenous labor case—condition (47)—by θ_{CD} , it follows from (89) that $\tilde{\theta}_{CD}/\theta_{CD} = \eta$, which, by assumption is less than one.

Conversely, in a stable labor share equilibrium, consumption, capital income, and labor income must grow at the same rate. Combining the condition on wage growth in a stable labor share equilibrium, (9), with intertemporal optimality gives, like before, that

$$\bar{F}_K = \delta_o + \rho + \frac{\theta}{1 - \sigma}$$

and this the stable labor share condition that $\bar{F}_K < A_K$ becomes

$$\theta < \eta(1 - \sigma)(A_K - \delta_o - \rho) \quad (90)$$

completing the proof.