Ordering information content using the quantile function*

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Abstract

Lehmann's (1988) ordering of information content is equivalent to a single-crossing property of the quantile function. This equivalence considerably aids the application of Lehmann's ordering.

The information content of prices and agents' actions is central to many areas of economics. Blackwell (1953) develops a very general notion of information content: a random variable X is more informative than a random variable Y if a decisionmaker would prefer to observe X than Y, regardless of the decision problem faced. However, Blackwell's ordering fails to rank many cases of interest. For example, Lehmann (1988) shows that, surprisingly, Blackwell's ordering fails to rank the amount of information about a variable θ that is is conveyed by the family of random variables $X_\kappa = \theta + \frac{\nu}{\kappa}$, where ν is uniformly distributed over $[-1, 1]$.

Lehmann (1988) proposes an alternative notion of information content that ranks more cases than Blackwell's, including the example just given. Lehmann's ordering is stated in a way that is likely to be intuitive to economists: rather than insisting that an arbitrary decisionmaker prefer to observe X rather than Y, Lehmann considers only monotone decision problems, i.e., those in which the decision of a fully informed decisionmaker would be monotone in the underlying state variable. Nonetheless, Lehmann's ordering has attracted

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relatively limited attention from economists.[1] Moreover, Lehmann's ordering is stated in a way that, at first sight, it hard operationalize.

The main result in this short paper is that Lehmann's ordering is equivalent to a single-crossing property of the quantile function. Although simple, this equivalence has not previously been noted. Exploiting this equivalence considerably aids the application of Lehmann's ordering, especially in differentiable cases. In particular, by working with the Spence-Mirrlees formulation of single-crossing, it is possible to compare the information content of prices and actions without fully solving for prices or optimal actions.

1 Lehmann-informativeness and the single-crossing property of the quantile function

Lehmann's ordering is as follows. A decisionmaker observes $X \in \Re$. The distribution of X depends on some underlying random variable, $\theta \in \Theta$, that is relevant to to decisionmaker, but that he cannot observe. The distribution of X also depends on a "regime," which is indexed by the parameter κ. The interpretation of the regime κ depends on the application. In the simple case above of $X = \theta + \frac{\nu}{\kappa}$, the regime κ is simply a scaling parameter that controls the variance of the "noise" term $\frac{\nu}{\kappa}$. In applications below, the regime κ will relate to the success probability of a project, and to frictions associated with taking a short position in a security.

Let $F(\cdot|\theta;\kappa)$ be the distribution of X conditional on θ in regime κ, with $\mathcal{X}(\theta;\kappa)$ denoting the corresponding support. Define $\mathcal{X}(\kappa) \equiv \bigcup_{\theta\in\Theta} \mathcal{X}(\theta;\psi)$. For any $x \in \mathcal{X}(\kappa)$, define $\Theta(x;\kappa)$ as the set of states such that x lies in the support of X, i.e., $\Theta(x;\kappa) = \left\{\tilde{\theta} : x \in \mathcal{X}\left(\tilde{\theta};\kappa\right)\right\}$.

Lehmann's ordering compares how much information X conveys about θ in two alternate regimes κ_1 and κ_2. Define the function $I(\cdot,\theta;\kappa_1,\kappa_2) : \mathcal{X}(\theta;\kappa_2) \to \mathcal{X}(\theta;\kappa_1)$ by

$$F\left(I\left(x,\theta;\kappa_1,\kappa_2\right)|\theta;\kappa_1\right) = F\left(x|\theta;\kappa_2\right).$$

Definition 1 *X is a more Lehmann-informative in regime κ_2 than κ_1 if for all $x \in \mathcal{X}(\kappa_2)$, the function $I(x,\theta;\kappa_1,\kappa_2)$ is weakly decreasing in $\theta \in \Theta(x;\kappa_2)$.*

Remark: Definition 1 differs slightly from Lehmann's original definition, which is that the inverse of I with respect to x is weakly increasing in θ.[2] Under mild regularity conditions on

[1]For significant exceptions, see Jewitt (2007), Athey and Levin (2018), Perisco (2000), Bergemann and Valimaki (2002), and Quah and Strulovici (2009).

[2]The inverse of I is the function $J(\cdot,\theta;\kappa_1,\kappa_2) : \mathcal{X}(\theta;\kappa_1) \to \mathcal{X}(\theta;\kappa_2)$ defined by $F(x|\theta;\kappa_1) = F(J(x,\theta;\kappa_1,\kappa_2)|\theta;\kappa_2)$.

how the support $\mathcal{X}(\theta;\kappa)$ varies with the underlying state θ, the two conditions are equivalent (see online appendix). Definition 1 has the advantage of being the condition that is used in the proof of Proposition 1, and avoids the need to impose further conditions on how the support $\mathcal{X}(\theta;\kappa)$ varies with θ.

1.1 Lehmann-informativeness and decision problems

Lehmann-informativeness is of interest because it ranks outcomes in a particular class of decision problems. Specifically, consider a decisionmaker who must select $b \in B \subset \Re$. The decisionmaker's objective is to choose b to maximize an objective $V(b,\theta)$, which is continuous in b. The decisionmaker does not observe θ directly, and instead observes only X, as described above.

The objective V satisfies the single-crossing property (SCP, Milgrom and Shannon (1994)) in (b,θ). Hence the decision problem is monotone, in the sense that a decisionmaker who were counterfactually fully informed about θ would choose higher values of b when θ is higher.

To allow for cases in which the choice set B in non-compact, I impose the following relatively mild assumption on how V behaves for low and high choices of $b \in B$: There exist $\underline{\theta}$, $\bar{\theta} \geq \underline{\theta}$, $\underline{b}$ and $\bar{b}$ such that if $\theta \leq \underline{\theta}$ then $V(\cdot,\theta)$ is weakly decreasing for $b \geq \bar{b}$, and if $\theta \geq \bar{\theta}$ then $V(\cdot,\theta)$ is weakly increasing for $b \leq \underline{b}$.

Lehmann (1988) and Quah and Strulovici (2009) establish that if X is more Lehmann-informative in regime κ_2 than κ_1, then the decisionmaker is better off in regime κ_2 than in κ_1.[3] Both papers restrict attention to the case in which the support of X is independent of the realization of θ. To facilitate applications, Proposition 1 below represents a modest generalization of these previous results to the case in which the support of X potentially depends on θ, and in which the decisionmaker's action space is non-compact. For convenience, I state the following two properties used in Proposition 1 separately.

First, the distribution of X in state θ admits a density:

Property 1 *For all states θ and regimes κ, the support $\mathcal{X}(\theta;\kappa)$ is an interval, and the distribution function $F(\cdot|\theta;\kappa)$ is continuous and strictly increasing over $\mathcal{X}(\theta;\kappa)$, with* $\inf_{x\in\mathcal{X}(\theta;\kappa)} F(x|\theta;\kappa) = 0$ *and* $\sup_{x\in\mathcal{X}(\theta;\kappa)} F(x|\theta;\kappa) = 1$.

Second, any shift in the support of X across regimes satisfies the following mild restriction:

Property 2 *Let $\theta \in \Theta$ and $\kappa, \tilde{\kappa}$ be alternate regimes. The support $\mathcal{X}(\theta;\kappa)$ is unbounded above (respectively, below) if and only if $\mathcal{X}(\theta;\tilde{\kappa})$ is unbounded above (below).*

[3] Lehmann (1988) imposes slightly different assumptions on the decisionmaker's objective.

Proposition 1 *Let Properties 1 and 2 hold. If X is more Lehmann-informative in regime κ_2 than κ_1, and $\zeta : \mathcal{X}(\kappa_1) \to B$ is a weakly increasing function, then there exists $\phi : \mathcal{X}(\kappa_2) \to B$ such that, for all θ, $V(\phi(X), \theta)$ in regime κ_2 first-order stochastically dominates $V(\zeta(X), \theta)$ in regime κ_1.*

Because Proposition 1 is close to existing results, I relegate its proof to Appendix B.

As Quah and Strulovici (2009) emphasize, Lehmann-informativeness implies an improvement in the uninformed investor's payoff in a very robust sense, in that Proposition 1 is completely independent of the uninformed investor's prior beliefs of θ.

Proposition 1 is predicated on the decisionmaker's action being weakly increasing in X in the initial regime κ_1. Lehmann (1988) and Quah and Strulovici (2009) each give sufficient conditions for this. In both cases, the conditions include that the monotone likelihood ratio property (MLRP) holds.[4]

1.2 Equivalence of Lehmann-informativeness to the quantile function satisfying the SCP

The main result in this paper is that Lehmann-informativeness is equivalent to a single-crossing property of the quantile function $F^{-1}(1-t|\theta;\kappa)$, which gives the value of x associated with X lying in the top t percentile.

Given Property 1, for all $t \in (0,1)$ the quantile function $F^{-1}(1-t|\theta;\kappa)$ is uniquely defined. In addition, define $F^{-1}(0|\theta;\psi) = \inf \mathcal{X}(\theta;\psi)$ and $F^{-1}(1|\theta;\psi) = \sup \mathcal{X}(\theta;\psi)$, with the understanding that if $\mathcal{X}(\theta;\psi)$ is unbounded below (respectively, above) then $\inf \mathcal{X}(\theta;\psi) = -\infty$ (respectively, $\sup \mathcal{X}(\theta;\psi) = \infty$).

The equivalence of Lehmann-informativeness with the quantile function satisfying the SCP naturally requires an ordering on the set of states Θ. The standard first-order stochastic dominance (FOSD) ordering is sufficient. Because of the centrality of the quantile function to the analysis, I include an equivalent formulation of FOSD in terms of the quantile function[5] in the following:

Property 3 *For any regime κ, if $\theta_2 > \theta_1$ then the distribution of X given θ_2 FOSD the distribution of X given θ_1, i.e., $F(x|\theta_2;\kappa) \leq F(x|\theta_1;\kappa)$ for any x, or equivalently, $F^{-1}(1-t|\theta_2;\kappa) \geq F^{-1}(1-t|\theta_1;\kappa)$.*

[4] That is: $\frac{f(x|\theta_2;\kappa)}{f(x|\theta_1;\kappa)}$ is weakly increasing in x if $\theta_2 > \theta_1$, where $f(x|\theta;\kappa)$ denotes the density function corresponding to the distribution function $F(x|\theta;\kappa)$.

[5] See, for example, Theorem 4.1 in Levy (1998).

As noted, Lehmann (1988) and Quah and Strulovici (2009) both impose MLRP, which implies FOSD.[6]

The main result is:

Proposition 2 *Let Properties 1-3 hold. The Lehmann-informativeness of X is increasing in the regime κ if and only if the quantile function $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta \times [0,1]$ has the product ordering.*

The following simple example illustrates Proposition 2.

Example: Let $X = \theta + \frac{\nu}{\kappa}$, where ν is distributed uniformly over $[-1,1]$. Theorem 5.3 in Lehmann establishes that Lehmann informativeness is increasing in κ.[7] The quantile function is $F^{-1}(1-t|\theta;\kappa) = \theta + \frac{1-2t}{\kappa}$. Hence if $t_2 \geq t_1$, $\theta_2 \geq \theta_1$, and $\kappa_2 \geq \kappa_1$, then $F^{-1}(1-t_2|\theta_2;\kappa_1) \geq (>) F^{-1}(1-t_2|\theta_2;\kappa_1)$ is equivalent to $\theta_2 - \theta_1 \geq (>) \frac{2t_2-2t_1}{\kappa_1}$, which implies that $F^{-1}(1-t_2|\theta_2;\kappa_2) \geq (>) F^{-1}(1-t_2|\theta_2;\kappa_2)$. Hence $F^{-1}(1-t|\theta;\kappa)$ indeed satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta \times [0,1]$ has the product ordering.

Graphically, the SCP corresponds to the isoquants of the quantile function $F^{-1}(1-t|\theta;\kappa)$ in (θ,t) growing steeper as κ increases (see Figure 1).[8] Intuitively, steeper isoquants correspond to greater information content, as follows. The observation of a realization x of X has the same information content as the observation of what quantile x belongs to. Steeper isoquant curves of the quantile function correspond to the quantile containing a lot of information about the value of θ.

1.3 Spence-Mirrlees single-crossing

To check whether the quantile function $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP, it is useful to relate it to the Spence-Mirrlees single-crossing condition, which is expressed in terms of derivatives. Milgrom and Shannon's (1994) Theorem 3 establishes the equivalence (under certain conditions) between the Spence-Mirrlees condition and the SCP under the lexicographic ordering. Under Property 1, $F^{-1}(1-t|\theta;\kappa)$ is strictly decreasing in t, and under Property 3, $F^{-1}(1-t|\theta;\kappa)$ is weakly increasing in θ. Under these conditions, it is straightforward to show that the SCP under the product ordering coincides with the SCP under the lexicographic ordering, which in turn coincides with the Spence-Mirrlees condition.

Proposition 3 *Let Properties 1-3 hold.*

[6] See, for example, Gollier (2001).

[7] Evaluating, $I(x,\theta;\kappa_1,\kappa_2) = \frac{\kappa_2}{\kappa_1}(x-\theta)+\theta$ (regardless of the distribution of ν).

[8] In the example, the isoquants are given by $t = \frac{1}{2}(1+\kappa\theta-\kappa x)$.

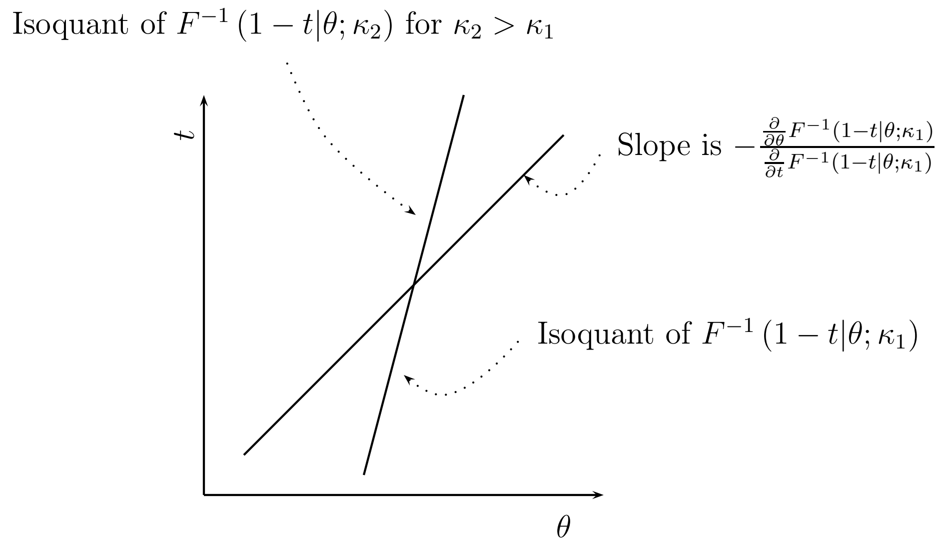

Figure 1: Graphical illustration of Lehmann-informativeness and SCP of the quantile function. Steeper isoquants for κ_2 than for κ_1 correspond to greater Lehmann-informativeness.

(I) $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta\times[0,1]$ has the product ordering, if and only if $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP in $((\theta,t);\kappa)$, where $\Theta\times[0,1]$ has the lexicographic ordering.

(II) If, moreover, $F^{-1}(1-t|\theta;\kappa)$ is differentiable with respect to θ and t, with derivatives continuous in (θ,t,κ), then the Lehmann-informativeness of X is increasing in the regime κ if and only if $F^{-1}(1-t|\theta;\kappa)$ satisfies the Spence-Mirrlees single-crossing condition condition, i.e.,

$$\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|} \text{ is increasing in } \kappa. \qquad (1)$$

2 Applications

2.1 Learning from the actions of others

In many cases, economic agents learn from the actions of others. For example, one economic agent—the decisionmaker, in the formalism of this paper—may learn about the quality of an investment project by observing a second economic agent's willingness to invest funds in a similar project, where the second economic agent knows θ.

Specifically, suppose that the decisionmaker observes an investment x made by an investor, where the investor chooses the investment x to solve

$$\max_{x\geq 0}\left(1-\psi\left(\theta,\kappa\right)\right)\bar{u}\left(x;t\right)+\psi\left(\theta,\kappa\right)\underline{u}\left(x;t\right), \tag{2}$$

where $\bar{u}$ and $\underline{u}$ are both concave in x, with $\bar{u}_x > 0 > \underline{u}_x$. That is, there are "success" and "failures" outcomes, where failure occurs with probability $\psi\left(\theta,\kappa\right)$, and depends on the combination of a state variable θ and a regime κ. For example, κ could be a macroeconomic state, or a government policy. The investment x is beneficial conditional on success, but costly conditional on failure. Finally, t represents idiosyncratic factors that affect the investor's willingness to invest, such as the investor's wealth level, or the investor's exposure to other risks. Assume that $t\sim U(0,1)$. From the perspective of the decisionmaker, t introduces noise into the investor's decision. Finally, note that "investment" and the "investor" can be broadly interpreted in a number of different ways.

It is often suggested that investors pay more attention to fundamentals—here, θ—if they are more exposed to failure risk. Below, I formalize this idea by deriving a simple condition for when greater exposure to failure risk increases Lehmann-informativeness. In what is close to a normalization, I parameterize ψ so that $\psi_\kappa > 0$ and $\psi_\theta < 0$, i.e., higher values of κ correspond to higher failure probabilities, and higher values of θ correspond to better fundamentals (lower failure probabilities).

The assumptions on $\bar{u}$ and $\underline{u}$ imply

$$\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x\left(x;t\right)}{\underline{u}_x\left(x;t\right)}\right)<0. \tag{3}$$

In addition, suppose that, for all values of θ, t, κ,

$$\left(1-\psi\left(\theta,\kappa\right)\right)\bar{u}_x\left(0;t\right)+\psi\left(\theta,\kappa\right)\underline{u}\left(0;t\right)>0,$$

so that investment is always positive; and that the idiosyncratic factor t affects the utility ratio $-\frac{\bar{u}_x}{\underline{u}_x}$ according to

$$\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x\left(x;t\right)}{\underline{u}_x\left(x;t\right)}\right) \;<\; 0, \tag{4}$$

$$\frac{\partial^2}{\partial t\partial x}\ln\left(-\frac{\bar{u}_x\left(x;t\right)}{\underline{u}_x\left(x;t\right)}\right) \;\leq\; 0. \tag{5}$$

To interpret (4), suppose that t is an underlying characteristic that raises the marginal utilities $\bar{u}_x$ and $-\underline{u}_x$. Then (4) says that t raises marginal utility proportionally more in the "bad" state than in the "good" state. This is a natural property. For example, it is satisfied in the following two specifications, which correspond to the idiosyncratic term t representing the investor's wealth level and risk exposure, respectively: (I) $\bar{u}_x(x;t) = u\left(W(t) + \bar{R}x\right)$ and $\underline{u}_x(x;t) = u\left(W(t) + \underline{R}x\right)$, (II) $\bar{u}_x(x;t) = (1-t)\,u\left(\bar{W} + \bar{R}x\right) + tu\left(\underline{W} + \bar{R}x\right)$ and $\underline{u}_x(x;t) = (1-t)\,u\left(\bar{W} + \underline{R}x\right) + tu\left(\underline{W} + \underline{R}x\right)$, where in both specifications $\underline{R} < 0 < \bar{R}$, and u features decreasing absolute risk aversion (DARA); in (I), $W'(t) < 0$, and in (II), $\underline{W} < \bar{W}$.[9]

Inequality (5) is essentially a regularity condition, and says that the extent to which t raises marginal utility proportionally more in the "bad" state than in the "good" state increases as the investor invests more. It is satisfied by both specifications (I) and (II).

Let $x(\theta, t, \kappa)$ denote the investor's optimal investment. Given concavity, it is determined by the first-order condition (FOC) of (2), which can be straightforwardly written as

$$\ln\left(-\frac{\bar{u}_x\left(x(\theta,t,\kappa);t\right)}{\underline{u}_x\left(x(\theta,t,\kappa);t\right)}\right) + \ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right) = 0. \tag{6}$$

Differentiation of (6) with respect to θ and t delivers

$$\begin{aligned}
x_\theta(\theta,t,\kappa)\,\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Bigg|_{x=x(\theta,t,\kappa)} &= -\frac{\partial}{\partial\theta}\ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right)\\
x_t(\theta,t,\kappa)\,\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Bigg|_{x=x(\theta,t,\kappa)} &= -\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Bigg|_{x=x(\theta,t,\kappa)}\\
x_\kappa(\theta,t,\kappa)\,\frac{\partial}{\partial x}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Bigg|_{x=x(\theta,t,\kappa)} &= -\frac{\partial}{\partial\kappa}\ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right).
\end{aligned}$$

From (3), it follows that $x_\theta > 0$, and, from (4), that $x_t < 0$. So Properties 1-3 are satisfied, and the quantile function of X is given by

$$F^{-1}(1-t|\theta;\kappa) = x(\theta,t,\kappa). \tag{7}$$

Moreover,

$$\frac{\frac{\partial}{\partial\theta}F^{-1}(1-t|\theta;\kappa)}{\left|\frac{\partial}{\partial t}F^{-1}(1-t|\theta;\kappa)\right|} = -\frac{x_\theta(\theta,t,\kappa)}{x_t(\theta,t,\kappa)} = \frac{\frac{\partial}{\partial\theta}\ln\left(\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}\right)}{-\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right)\Big|_{x=x(\theta,t,\kappa)}}. \tag{8}$$

Recall that $\psi_\kappa < 0$, i.e., as κ increases, the investor is more exposed to failure risk. Hence $x_\kappa < 0$, and so by (5) the denominator in (8) is decreasing in κ.

The ratio $\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}$ is the likelihood ratio of good and bad outcomes for the investor. So by Proposition 3, a sufficient condition for the Lehmann-informativeness of the investment x to

[9] The online appendix contains a proof that both specifications (I) and (II) satisfy both (4) and (5).

increase in exposure to failure risk (κ) is that the likelihood ratio $\frac{1-\psi(\theta,\kappa)}{\psi(\theta,\kappa)}$ be log supermodular. To interpret this condition, recall that the ratio $\frac{1-\psi}{\psi}$ is increasing in θ. So log supermodularity says that the likelihood ratio $\frac{1-\psi}{\psi}$ becomes more sensitive to θ as κ increases.[10]

2.2 Learning from prices

Instead of a decisionmaker learning from the actions of others, as in the previous subsection, I consider now the case of a decisionmaker who learns from prices. To give a specific example, a lender may seek to learn the riskiness of mortgage lending from the traded prices of mortgage-backed securities. Bond, Edmans, and Goldstein (2012) survey the literature on agents learning from financial prices.

Since the price is the object the decisionmaker is learning from, I denote the price by x. The price is determined by the standard market clearing conditions. Specifically, the market is populated by a mixture of informed agents, who observe θ and choose a quantity q to maximize utility $U(q,x,\theta,\kappa)$, and other agents, who trade for idiosyncratic reasons unrelated to either the price x or the state θ. These agents are analogous to the "noise" or "liquidity" traders in Grossman and Stiglitz (1980) and a large subsequent literature. Let the excess demand stemming from these noise traders be $-s'(t)$, where $t \sim U(0,1)$ and $s'(t) > 0$. Moreover, I assume that $s(0)$ and $s(1)$ are both finite, and focus on the case in which $s(0)$ and $s(1)$ are sufficiently small, as explained below. The utility function of informed agents takes the form

$$U(q,x,\theta,\kappa) = (1-\psi(\theta,\kappa))\, u\left(q\left(\bar{R}-x\right)\right) + \psi(\theta,\kappa)\, u\left(q\left(\underline{R}-x\right)\right),$$

where u is increasing and concave, $\bar{R} > \underline{R} \geq 0$, and ψ is again parameterized so that $\psi_\kappa > 0$ and $\psi_\theta < 0$. Note that this specification falls outside the CARA-normal framework that is used in many noisy rational expectation models.[11]

An informed agent's demand at price x is given by $q(x,\theta,\kappa)$, and is determined by the FOC

$$U_q\left(q(x,\theta,\kappa), x, \theta, \kappa\right) = 0. \tag{9}$$

The equilibrium price $x(\theta,t,\kappa)$ is then determined by the market clearing condition

$$q\left(x(\theta,t,\kappa), \theta, \kappa\right) = s(t). \tag{10}$$

[10] As a simple example, suppose that $\psi = (1-\theta)\kappa$, which can be interpreted as saying that failure occurs only if both a project specific event occurs (probability $1-\theta$) *and* an external event occurs (probability κ). Note that $\frac{\partial}{\partial\theta}\ln\frac{1-\psi}{\psi} = \frac{\kappa}{1-(1-\theta)\kappa} + \frac{1}{1-\theta}$, which is increasing in κ, i.e., log supermodularity holds.

[11] A recent exception is Breon-Drish (2015), and in common with that paper, the analysis below exploits market clearing conditions heavily.

Assume that $\psi \in (0,1)$ for all values of θ and κ. Then an immediate consequence of market clearing (10) is that the price x lies in the open interval $\left(\underline{R}, \bar{R}\right)$. Hence $U_{q\theta} > 0$, $U_{q\kappa} < 0$, in addition to $U_{qq} < 0$ and $U_{qx} > 0$.

As in the preceding subsection, I parameterize ψ so that $\psi_\theta < 0$ and $\psi_\kappa > 0$, i.e., higher values of θ are better for informed agents (holding prices fixed), and higher values of κ correspond to agents facing a higher probability of the bad realization, $\underline{R}$.

Differentiation of the FOC for informed demand (9) yields

$$\begin{aligned} q_x U_{qq}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) + U_{qx}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) &= 0, \\ q_\theta U_{qq}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) + U_{q\theta}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) &= 0, \\ q_\kappa U_{qq}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) + U_{q\kappa}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right) &= 0. \end{aligned}$$

Hence quantity demanded is decreasing in price x, increasing in θ, and decreasing in κ, as one would expect: $q_x < 0$, $q_\theta > 0$, and $q_\kappa < 0$. Differentiation of the market clearing (10) condition yields

$$\begin{aligned} x_\theta q_x + q_\theta &= 0 \\ x_t q_x - s'\left(t\right) &= 0 \\ x_\kappa q_x + q_\kappa &= 0. \end{aligned}$$

So $x_\theta > 0$ and $x_t < 0$, implying that the quantile function simply equals the price, i.e., (7) holds, and that Properties 1-3 are satisfied. Moreover,

$$\frac{\frac{\partial}{\partial\theta}F^{-1}\left(1-t|\theta;\kappa\right)}{\left|\frac{\partial}{\partial t}F^{-1}\left(1-t|\theta;\kappa\right)\right|} = -\frac{x_\theta}{x_t} = \frac{q_\theta}{s'\left(t\right)}.$$

Hence, by Proposition 3, to evaluate Lehmann-informativeness one must sign

$$\frac{\partial}{\partial\kappa}\left(\frac{\frac{\partial}{\partial\theta}F^{-1}\left(1-t|\theta;\kappa\right)}{\left|\frac{\partial}{\partial t}F^{-1}\left(1-t|\theta;\kappa\right)\right|}\right) = \frac{q_{\theta\kappa}+q_{\theta x}x_\kappa}{s'\left(t\right)} = \frac{q_{\theta\kappa}-\frac{q_{\theta x}q_\kappa}{q_x}}{s'\left(t\right)} = -\frac{q_\kappa}{s'\left(t\right)}\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\bigg|_{x=x(\theta,t,\kappa)}. \quad (11)$$

Expression (11) states the Spence-Mirrlees condition in terms of the demand function q, and so can be checked without solving explicitly for prices x. Moreover, further straightforward substitution relates the properties of demand directly to the utility function U: see online appendix for details, both here and below.

Straightforward but tedious algebra implies that there is a function $C\left(t\right)$ such that $C\left(t\right)$

converges uniformly to 0 as $s(0)$ and $s(1)$ approach 0, and such that

$$\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\Big|_{x=x(\theta,t,\kappa)} \geq -\psi_\theta\left(\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa}-\frac{1}{\psi}+C(t)\right). \tag{12}$$

Here, the term $\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa}-\frac{1}{\psi}$ stems from differentiating q_κ with respect to θ, and hence captures the interaction of θ and κ in determining demand q. Intuitively, this is the key determinant of whether the information content of the price is increasing in κ.

The term $C(t)$ stems from differentiating q_x with respect to θ. Intuitively, the interaction of θ and the price x in determining demand q stems from variation in an informed agent's wealth as x varies. Focusing on the case in which $s(0), s(1)$ are both small ensures that this effect is of second-order importance, since in this case informed agents' equilibrium holding of the asset is necessarily small, and hence that the dominant effect stems from the interaction of θ and κ.

Consequently, (12) implies that the Lehmann-informativeness of prices is increasing in κ if $\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} > \frac{1}{\psi}$ (recall that $\psi_\theta < 0$, $q_\kappa < 0$, $s'(t) > 0$), or equivalently, if $\psi_{\kappa\theta}\psi - \psi_\theta\psi_\kappa < 0$, i.e., if the probability ψ of the low realization $\underline{R}$ is strictly log-submodular in (θ, κ). To interpret this condition, recall that the "failure" probability ψ is decreasing in θ. So log-submodularity says that the failure probability becomes more sensitive to θ as κ increases.[12]

2.3 Trading constraints and the informational content of prices

In the two applications above, the regime κ determines the sensitivity of good and bad outcomes to the underlying state θ. Here, the regime κ instead determines trading frictions, and in particular, the possibility that an asset may be hard to short sell.

Various authors have argued that short-sales constraints prevent asset prices impounding information (see, e.g., Miller (1977), Hong and Stein (2007)). Below, I formalize this idea.

For maximal transparency, I adopt a very simple model. Informed investors know θ, have mean-variance preferences, and trade the asset. As before, $q(x, \theta, \kappa)$ denotes informed agent demand given asset price x, state θ, and regime κ. The regime κ affects shorting costs: positions $q < 0$ incur a per-unit cost of $K(\kappa)$, where without loss K is a decreasing function of the regime κ. So under mean-variance preferences, informed agent demand takes

[12] Log-submodularity of ψ implies log-supermodularity of the likelihood ratio $\frac{1-\psi}{\psi}$. The online appendix contains a proof.

the simple form

$$q(x,\theta,\kappa) = \begin{cases} \theta - x & \theta > x \\ 0 & \theta \in [x-K, x] \\ \theta - x + K & \theta < x - K \end{cases}, \tag{13}$$

where for expositional transparency I have normalized the residual variance and risk aversion parameters so that there is no further multiplicative constant. As is widely appreciated, transaction costs lead to a kinked demand function, with an interval of prices over which, for a given θ, informed agents do not want to take either long or short positions in the the asset, i.e., have locally perfectly inelastic demand.

There are also "noise" or "liquidity" traders in the market. I assume that these traders are price responsive, i.e., their demand depends on the price x.[13] Specifically, the demand of these traders is given by

$$-s(t) - \lambda x, \tag{14}$$

where $\lambda > 0$ is a parameter that determines price-sensitivity, and s is increasing in t.

The market-clearing condition is consequently

$$q(x,\theta,\kappa) = s(t) + \lambda x.$$

In contrast to the applications to subsections 2.1 and 2.2, the equilibrium value of x (here, price) can be solved for analytically, and takes the following simple form (here and below, all details are relegated to the online appendix):

$$x(\theta,t,\kappa) = \begin{cases} \frac{\theta - s(t)}{\lambda+1} & \text{if } \theta > -\frac{s(t)}{\lambda} \\ -\frac{s(t)}{\lambda} & \text{if } \theta \in \left[-\frac{s(t)}{\lambda} - K, -\frac{s(t)}{\lambda}\right] \\ \frac{\theta - s(t) + K}{\lambda+1} & \text{if } \theta < -\frac{s(t)}{\lambda} - K \end{cases}. \tag{15}$$

Again as one would expect, shorting costs generate an interval of fundamentals over which the price is independent of the fundamental.

Note that $x(\theta,t,\kappa)$ is increasing in θ, which ensures that Properties 1-3 are satisfied; and is strictly decreasing in t, so that the quantile function simply equals the price, i.e., (7) holds.

The price (15) does not satisfy the differentiability requirements needed to apply Proposition 3. But it is nonetheless straightforward to show that the price x satisfies SCP in $((\theta,t);\kappa)$. By Proposition 2, it follows that Lehmann-informativeness is increasing in κ, i.e.,

[13] If instead noise traders had perfectly inelastic demand, there are potentially cases in which both noise and informed traders have perfectly inelastic demand, complicating the analysis.

is decreasing in the size of the shorting cost K.

2.4 Lehmann informativeness and conditional variance

Subsections 2.2 and 2.3 analyze the information content of prices. In many studies of financial markets, the information content of prices is measured by their ability to predict future cash flows, as measured using second moments. In the notation of this paper, this boils down to the residual variance $var(\theta|X)$, i.e., θ is a future cash flow, X is the current price, and greater residual variance corresponds to lower informativeness.[14]

Lehmann-informativeness implies the residual variance ordering, as follows. Suppose that X_2 is more Lehmann-informative than X_1. By Theorem 1(i) of Ganuza and Penalva (2010), it follows that $E\left[E[\theta|X_2]^2\right] \geq E\left[E[\theta|X_1]^2\right]$. By the law of total expectation, $E[E[\theta|X_2]] = E[E[\theta|X_1]] = E[\theta]$. Consequently, $var(E[\theta|X_2]) \geq var(E[\theta|X_1])$. By the law of total variance, it follows that $E[var(\theta|X_2)] \leq E[var(\theta|X_1)]$, i.e., X_2 is more informative under the residual variance ordering.

3 Conclusion

Lehmann's (1988) information ordering is equivalent to a single-crossing property of the quantile function. Under mild differentiability conditions, Lehmann's ordering is also equivalent to Spence-Mirrlees single-crossing. These equivalences, which have not previously been noted, considerably aid the application of Lehmann's ordering.

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[14] See, for example, Brunnermeier (2005), Peress (2010), Bai et al (2016). Closely related are Dávila and Parlatore (2018), who define price informativeness using $var(X|\theta)$, i.e., the residual variance of current prices conditional on future cash flow innovations.

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A Proofs of Propositions 2 and 3

Proof of Proposition 2:

SCP implies Lehmann-informativeness: Fix κ_1, $\kappa_2 > \kappa_1$, $x \in \mathcal{X}(\kappa_2)$ and $\theta_1, \theta_2 \in$

$\Theta(x;\kappa_2)$. Let t_1 and t_2 be such that

$$F^{-1}(1-t_1|\theta_1;\kappa_2) = F^{-1}(1-t_2|\theta_2;\kappa_2) = x. \tag{16}$$

Hence

$$\begin{aligned} F(x|\theta_1;\kappa_2) &= 1-t_1 & (17)\\ F(x|\theta_2;\kappa_2) &= 1-t_2. & (18) \end{aligned}$$

By FOSD, it follows that $t_2 \geq t_1$. Equations (16), (17) and (18) also deliver

$$\begin{aligned} F\left(F^{-1}(1-t_1|\theta_1;\kappa_1)|\theta_1;\kappa_1\right) &= 1-t_1 = F(x|\theta_1;\kappa_2)\\ F\left(F^{-1}(1-t_2|\theta_2;\kappa_1)|\theta_2;\kappa_1\right) &= 1-t_2 = F(x|\theta_2;\kappa_2). \end{aligned}$$

Hence

$$\begin{aligned} I(x,\theta_1;\kappa_1,\kappa_2) &= F^{-1}(1-t_1|\theta_1;\kappa_1)\\ I(x,\theta_2;\kappa_1,\kappa_2) &= F^{-1}(1-t_2|\theta_2;\kappa_1). \end{aligned}$$

Equality (16) and the SCP then imply

$$F^{-1}(1-t_2|\theta_2;\kappa_1) \leq F^{-1}(1-t_1|\theta_1;\kappa_1),$$

establishing the result.

Lehmann-informativeness implies SCP:

Suppose that, contrary to the claimed result, the SCP is violated, i.e., there exist t_1, $t_2 \geq t_1$, θ_1, $\theta_2 \geq \theta_1$, κ_1 and $\kappa_2 \geq \kappa_1$ such that either

$$\begin{aligned} F^{-1}(1-t_2|\theta_2;\kappa_1) &= F^{-1}(1-t_1|\theta_1;\kappa_1) & (19)\\ F^{-1}(1-t_2|\theta_2;\kappa_2) &< F^{-1}(1-t_1|\theta_1;\kappa_2), & (20) \end{aligned}$$

or

$$\begin{aligned} F^{-1}(1-t_2|\theta_2;\kappa_1) &> F^{-1}(1-t_1|\theta_1;\kappa_1) & (21)\\ F^{-1}(1-t_2|\theta_2;\kappa_2) &\leq F^{-1}(1-t_1|\theta_1;\kappa_2). & (22) \end{aligned}$$

By Property 3,

$$F^{-1}(1-t_1|\theta_1;\kappa_2) \leq F^{-1}(1-t_1|\theta_2;\kappa_2),$$

so that, regardless of whether (20) or (22) holds,

$$F^{-1}(1-t_2|\theta_2;\kappa_2) \leq F^{-1}(1-t_1|\theta_2;\kappa_2).$$

It follows from Property 1 that there exists $t_3 \in [t_1, t_2]$ such that

$$F^{-1}(1-t_3|\theta_2;\kappa_1) > F^{-1}(1-t_1|\theta_1;\kappa_1) \quad (23)$$
$$F^{-1}(1-t_3|\theta_2;\kappa_2) = F^{-1}(1-t_1|\theta_1;\kappa_2). \quad (24)$$

Let $x = F^{-1}(1-t_1|\theta_1;\kappa_2) \in \mathcal{X}(\theta_1;\kappa_2)$. By Property 2, (24) implies that $x \in \mathcal{X}(\theta_2;\kappa_2)$. From (24) and the definition of x

$$F(x|\theta_1;\kappa_2) = 1-t_1$$
$$F(x|\theta_2;\kappa_2) = 1-t_3.$$

By an identical argument to that used in the first half of the proof,

$$I(x,\theta_1;\kappa_1,\kappa_2) = F^{-1}(1-t_1|\theta_1;\kappa_1)$$
$$I(x,\theta_2;\kappa_1,\kappa_2) = F^{-1}(1-t_3|\theta_2;\kappa_1).$$

So the Lehmann-informativeness condition implies

$$F^{-1}(1-t_3|\theta_2;\kappa_1) \leq F^{-1}(1-t_1|\theta_1;\kappa_1),$$

contradicting (23) and completing the proof.

Proof of Proposition 3:

Part (I): If (θ_2, t_2) exceeds (θ_1, t_1) under the product order, it does so under the lexicographic order also. As such, it is immediate that if $F^{-1}(1-t|\theta;\kappa)$ satisfies the SCP under the lexicographic order, it does so under the product order also. To establish the opposite implication, consider (θ_1, t_1), (θ_2, t_2), κ_1, and κ_2 such that (θ_2, t_2) exceeds (θ_1, t_1) under the lexicographic order; $\kappa_2 > \kappa_1$; and $F^{-1}(1-t_1|\theta_1;\kappa_1) \leq F^{-1}(1-t_2|\theta_2;\kappa_1)$. The only non-trivial case to consider is that in which (θ_2, t_2) *does not exceed* (θ_1, t_1) under the product order, i.e., $\theta_1 < \theta_2$ and $t_2 < t_1$. In this case, Properties 1 and 3 imply that, for any κ, $F^{-1}(1-t_1|\theta_1;\kappa) < F^{-1}(1-t_2|\theta_2;\kappa)$, completing the proof.

Part (II): As noted in the main text, Part (II) is an application of Milgrom and Shannon's (1994) Theorem 3. To apply this result it is necessary to verify the condition that

$F^{-1}(1-t|\theta;\kappa)$ is completely regular, which, given that F^{-1} is weakly increasing in θ, is equivalent to checking that if

$$F^{-1}(1-t_1|\theta_1;\kappa) = F^{-1}(1-t_2|\theta_2;\kappa)$$

for some $\theta_2 > \theta_1$, then for any $\theta \in (\theta_1, \theta_2)$ there exists $t(\theta)$ continuous in θ such that

$$F^{-1}(1-t(\theta)|\theta;\kappa) = F^{-1}(1-t_1|\theta_1;\kappa). \tag{25}$$

This condition is indeed satisfied since

$$F^{-1}(1-t_1|\theta;\kappa) \geq F^{-1}(1-t_1|\theta_1;\kappa) = F^{-1}(1-t_2|\theta_2;\kappa) \geq F^{-1}(1-t_2|\theta;\kappa),$$

and hence (by continuity) there exists a unique $t(\theta)$ such (25) holds. Continuity follows since F^{-1} is continuous in (θ, t, κ).

B Proof of Proposition 1

The heart of proof of Proposition 1 is the following result, which generalizes Step 2 of Lemma 3 in Quah and Strulovici (2009) to the case in which the action space B is non-compact.

Lemma 1 *If $b(\theta)$ is a weakly decreasing function then there exists b^* such that $V(b^*, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta$.*

Proof of Lemma 1: Consider first the case in which $b(\cdot)$ takes only finitely many values. Hence there is finite partition $\{\Theta_k : k = 1, \ldots, K\}$ of Θ such that $b(\cdot)$ is constant over each partition element Θ_k, and every member of Θ_{k+1} exceeds every member of Θ_k. The proof establishes the slightly stronger result that there exists $b^* \in [b(\Theta_K), b(\Theta_1)]$ such that $V(b^*, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta$.

The proof is by induction. Suppose there exists $\tilde{b}_k \geq b(\Theta_k)$ such that $V(\tilde{b}_k, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \bigcup_{j \leq k} \Theta_j$. To establish the result, it is sufficient to establish the inductive step that there exists $\tilde{b}_{k+1} \geq b(\Theta_{k+1})$ such that $V(\tilde{b}_{k+1}, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \bigcup_{j \leq k+1} \Theta_j$. Define $\tilde{b}_{k+1}$ as the supremum of

$$\arg \max_{b \in [b(\Theta_{k+1}), \tilde{b}_k]} V(b, \sup \Theta_k).$$

So in particular, $V(\tilde{b}_{k+1}, \sup \Theta_k) \geq V(b(\Theta_{k+1}), \sup \Theta_k)$. Since b is constant over Θ_{k+1}, SCP implies $V(\tilde{b}_{k+1}, \theta) \geq V(b(\theta), \theta)$ for all $\theta \in \Theta_{k+1}$. Moreover, $V(\tilde{b}_{k+1}, \theta) \geq V(\tilde{b}_k, \theta)$ for all

$\theta \in \bigcup_{j \leq k} \Theta_j$, since if instead $V\left(\tilde{b}_k, \theta\right) > V\left(\tilde{b}_{k+1}, \theta\right)$ for some $\theta \in \bigcup_{j \leq k} \Theta_j$, SCP implies that $V\left(\tilde{b}_k, \sup \Theta_k\right) > V\left(\tilde{b}_{k+1}, \sup \Theta_k\right)$, which contradicts the definition of $\tilde{b}_{k+1}$. By supposition, it then follows that $V\left(\tilde{b}_{k+1}, \theta\right) \geq V\left(b\left(\theta\right), \theta\right)$ for all $\theta \in \bigcup_{j \leq k+1} \Theta_j$, establishing the inductive step and hence completing the proof of this case.

Next, consider the case in which $b\left(\cdot\right)$ take infinitely many values. Recall that $\underline{\theta}$, $\bar{\theta}$, $\underline{b}$ and $\bar{b}$ are defined in subsection 1.1. Define

$$\beta\left(\theta\right) = \begin{cases} \min\left\{b\left(\theta\right), \max\left\{b\left(\underline{\theta}\right), \bar{b}\right\}\right\} & \text{if } \theta \leq \underline{\theta} \\ b\left(\theta\right) & \text{if } \theta \in \left(\underline{\theta}, \bar{\theta}\right) \\ \max\left\{b\left(\theta\right), \min\left\{b\left(\bar{\theta}\right), \underline{b}\right\}\right\} & \text{if } \theta \geq \bar{\theta} \end{cases}.$$

Define $\bar{B} = \left[\min\left\{b\left(\bar{\theta}\right), \underline{b}\right\}, \max\left\{b\left(\underline{\theta}\right), \bar{b}\right\}\right]$. Observe that β is weakly decreasing and $\beta\left(\Theta\right) \subset \bar{B}$. Moreover, if $\beta\left(\theta\right) \neq b\left(\theta\right)$ then either $\theta \leq \underline{\theta}$ and $b\left(\theta\right) > \beta\left(\theta\right) \geq \bar{b}$, or $\theta \geq \bar{\theta}$ and $b\left(\theta\right) < \beta\left(\theta\right) \leq \underline{b}$. So by the definition of $\underline{\theta}$, $\bar{\theta}$, $\underline{b}$ and $\bar{b}$,

$$V\left(\beta\left(\theta\right), \theta\right) \geq V\left(b\left(\theta\right), \theta\right) \text{ for all } \theta \in \Theta. \tag{26}$$

Let $\{B_n\}$ be a sequence of finite subsets of $\bar{B}$ such that $B_n \subset B_{n+1}$ and $\bigcup_n B_n$ is dense in $\bar{B}$. Define $\beta_n\left(\theta\right)$ as the largest member of B_n that is weakly less than $\beta\left(\theta\right)$. Hence for any $\theta \in \Theta$, $\beta_{n+1}(\theta) \geq \beta_n\left(\theta\right)$ and $\beta_n\left(\theta\right) \to \beta\left(\theta\right)$.

For any n, the first part of the proof implies that there exists b_n^* such that $V\left(b_n^*, \theta\right) \geq V\left(\beta_n\left(\theta\right), \theta\right)$ for all $\theta \in \Theta$. Moreover, $b_n^* \in \bar{B}$. Hence b_n^* has a convergent subsequence, with limit b^*. By the continuity of V in its first argument, it follows that $V\left(b^*, \theta\right) \geq V\left(\beta\left(\theta\right), \theta\right)$ for all $\theta \in \Theta$. The result then follows from (26), completing the proof.

Proof of Proposition 1: Under Property 1, for any θ, $I\left(\cdot, \theta\right)$ is strictly increasing. Let $J\left(\cdot, \theta\right) : \mathcal{X}\left(\theta; \kappa_1\right) \to \mathcal{X}\left(\theta; \kappa_2\right)$ be the inverse of $I\left(\cdot, \theta\right)$ with respect to its first argument. By Properties 1 and 2, the function $J\left(\cdot; \theta\right)$ is well-defined, and is strictly increasing.

Note first that X in regime κ_2 and $J\left(X, \theta\right)$ in regime κ_1 have the same distribution, since for any $x \in \mathcal{X}\left(\theta; \kappa_2\right)$,

$$\begin{aligned} \Pr\left(X \leq x | \theta; \kappa_2\right) &= F\left(x | \theta; \kappa_2\right) \\ &= F\left(I\left(x, \theta\right) | \theta; \kappa_1\right) \\ &= \Pr\left(X \leq I\left(x, \theta\right) | \theta; \kappa_1\right) \\ &= \Pr\left(J\left(X, \theta\right) \leq J\left(I\left(x, \theta\right), \theta\right) | \theta; \kappa_1\right) \\ &= \Pr\left(J\left(X, \theta\right) \leq x | \theta; \kappa_1\right). \end{aligned}$$

By the Lehmann-informativeness property, for any $x \in \mathcal{X}(\kappa_2)$ the function $\zeta(I(x,\theta))$ is weakly decreasing in θ over $\Theta(x;\kappa_2)$. So by Lemma 1, there exists a function $\phi : \mathcal{X}(\kappa_2) \to B$ such that, for any $x \in \mathcal{X}(\kappa_2)$,

$$V(\phi(x),\theta) \geq V(\zeta(I(x,\theta)),\theta) \text{ for all } \theta \in \Theta(x;\kappa_2).$$

It follows that, for any θ and $\bar{V}$,

$$\begin{aligned} \Pr\left(V(\phi(X),\theta) \leq \bar{V}|\theta;\kappa_2\right) &= \Pr\left(V(\phi(J(X,\theta)),\theta) \leq \bar{V}|\theta;\kappa_1\right) \\ &\leq \Pr\left(V(\zeta(I(J(X,\theta),\theta)),\theta) \leq \bar{V}|\theta;\kappa_1\right) \\ &= \Pr\left(V(\zeta(X),\theta) \leq \bar{V}|\theta;\kappa_1\right), \end{aligned}$$

where the inequality uses $J(X,\theta) \in \mathcal{X}(\theta;\kappa_2)$, completing the proof.

Supplementary Online Appendix

B.1 Notes on the definition of Lehmann informativeness

I have defined Lehmann informativeness in terms of the function $I(x,\theta) : \mathcal{X}(\theta;\kappa_2) \to \mathcal{X}(\theta;\kappa_1)$, defined by

$$F(I(x,\theta)|\theta;\kappa_1) = F(x|\theta;\kappa_2).$$

The condition is:

[L-I] For any $x \in \mathcal{X}(\kappa_2)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1;\kappa_2) \cap \mathcal{X}(\theta_2;\kappa_2)$, $I(x,\theta_1) \geq I(x,\theta_2)$.

Typically, the definition is instead stated in terms of the function $J(x,\theta) : \mathcal{X}(\theta;\kappa_1) \to \mathcal{X}(\theta;\kappa_2)$, defined by

$$F(x|\theta;\kappa_1) = F(J(x,\theta)|\theta;\kappa_2).$$

The condition is then:

[L-J] For any $x \in \mathcal{X}(\kappa_1)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1;\kappa_1) \cap \mathcal{X}(\theta_2;\kappa_1)$, $J(x,\theta_2) \geq J(x,\theta_1)$.

Note that I and J are inverses. Specifically, for any $x \in \mathcal{X}(\theta;\kappa_1)$, $I(J(x,\theta)) = x$, and for any $x \in \mathcal{X}(\theta;\kappa_2)$, $J(I(x,\theta)) = x$. These statements make use of the fact that both I and J are strictly increasing in their first argument.

B.1.1 The advantage of stating the Lehmann informativeness in terms of [L-I]

The two formulations are equivalent under mild regularity conditions. The property actually used in the proof is that I is decreasing. Given non-equivalence under "pathological" conditions, it is easiest to simply state the definition in terms of [L-I].

B.1.2 Equivalence under many conditions

When the supports $\mathcal{X}(\theta;\kappa)$ are well-behaved, in terms of not varying too much in θ, the two definitions are equivalent.

Specifically:

Lemma 2 *If $\mathcal{X}(\theta_1;\kappa_1) \cap \mathcal{X}(\theta_2;\kappa_1) \neq \emptyset$ for all $\theta_1, \theta_2 \in \Theta$ then [L-J] implies [L-I].*

Lemma 3 *If $\mathcal{X}(\theta_1;\kappa_2) \cap \mathcal{X}(\theta_2;\kappa_2) \neq \emptyset$ for all $\theta_1, \theta_2 \in \Theta$ then [L-I] implies [L-J].*

Note, moreover, that the global non-empty intersection properties can be considerably weakened to ones that hold only local. For transparency, I state the proof for the global property.

Proof of Lemma 2: Suppose [L-J] holds, but [L-I] is violated, i.e., for some $x \in \mathcal{X}(\kappa_2)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2)$, $I(x, \theta_2) > I(x, \theta_1)$.

Certainly $I(x, \theta_1) \in \mathcal{X}(\theta_1; \kappa_1)$ and $I(x, \theta_2) \in \mathcal{X}(\theta_2; \kappa_1)$. Since $\mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1) \neq \emptyset$, it follows that there exists $x_0 \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$ such that

$$I(x, \theta_2) \geq x_0 \geq I(x, \theta_1),$$

with at least one of the two inequalities strict. But then

$$x = J(I(x, \theta_2), \theta_2) \geq J(x_0, \theta_2) \geq J(x_0, \theta_1) \geq J(I(x, \theta_1), \theta_1) = x,$$

with at least one of the first and third inequalities being strict. The contradiction completes the proof.

Proof of Lemma 3: Suppose [L-I] holds, but [L-J] is violated, i.e., for some $x \in \mathcal{X}(\kappa_1)$, and $\theta_1, \theta_2 > \theta_1$ such that $x \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$, $J(x, \theta_2) < J(x, \theta_1)$.

Certainly $J(x, \theta_1) \in \mathcal{X}(\theta_1; \kappa_2)$ and $J(x, \theta_2) \in \mathcal{X}(\theta_2; \kappa_2)$. Since $\mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2) \neq \emptyset$, it follows that there exists $x_0 \in \mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2)$ such that

$$J(x, \theta_2) \leq x_0 \leq J(x, \theta_1),$$

with at least one of the two inequalities strict. But then

$$x = I(J(x, \theta_2), \theta_2) \leq I(x_0, \theta_2) \leq I(x_0, \theta_1) \leq I(J(x, \theta_1), \theta_1) = x,$$

with at least one of the first and third inequalities being strict. The contradiction completes the proof.

B.1.3 A simple example in which [L-I] holds but [L-J] is violated

Consider a case in which $\Theta = \{\theta_1, \theta_2\}$, with $\theta_2 > \theta_1$, $\mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1) \neq \emptyset$ but $\mathcal{X}(\theta_1; \kappa_2) \cap \mathcal{X}(\theta_2; \kappa_2) = \emptyset$, and $\mathcal{X}(\theta_2; \kappa_2) < \mathcal{X}(\theta_1; \kappa_2)$. (Since these sets don't intersect, this ordering is unambiguous.)

In this case, [L-I] holds vacuously, while trivially, if $x \in \mathcal{X}(\theta_1; \kappa_1) \cap \mathcal{X}(\theta_2; \kappa_1)$, then $J(x, \theta_2) < J(x, \theta_1)$.

B.2 Detailed calculations used in subsection 2.1

Consider $\bar{u}_x(x;t) = u\left(W(t) + \bar{R}x\right)$ and $\underline{u}_x(x;t) = u\left(W(t) + \underline{R}x\right)$, where $W'(t) < 0$, $\underline{R} < 0 < \bar{R}$, and u features decreasing absolute risk aversion (DARA).

In this case,

$$\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right) = \frac{\partial}{\partial t}\ln\left(-\frac{\bar{R}}{\underline{R}}\frac{u'\left(W(t)+\bar{R}x\right)}{u'\left(W(t)+\bar{R}x\right)}\right) = W'(t)\frac{u''\left(W(t)+\bar{R}x\right)}{u'\left(W(t)+\bar{R}x\right)} - W'(t)\frac{u''\left(W(t)+\underline{R}x\right)}{u'\left(W(t)+\underline{R}x\right)}.$$

Hence DARA implies (4). Moreover, DARA further implies that $-\frac{u''\left(W+\bar{R}x-t\right)}{u'\left(W+\bar{R}x-t\right)}$ is decreasing in x and $-\frac{u''(W+\underline{R}x-t)}{u'(W+\underline{R}x-t)}$ is increasing in x, so that (5) holds.

Consider $\bar{u}_x(x;t) = (1-t)\,u\left(\bar{W}+\bar{R}x\right) + tu\left(\underline{W}+\bar{R}x\right)$ and $\underline{u}_x(x;t) = (1-t)\,u\left(\bar{W}+\underline{R}x\right) + tu\left(\underline{W}+\underline{R}x\right)$, where $\underline{W} < \bar{W}$, $\underline{R} < 0 < \bar{R}$, an u features DARA. In this case,

$$\begin{aligned}
\frac{\partial}{\partial t}\ln\left(-\frac{\bar{u}_x(x;t)}{\underline{u}_x(x;t)}\right) &= -\frac{u'\left(\underline{W}+\bar{R}x\right) - u'\left(\bar{W}+\bar{R}x\right)}{(1-t)\,u'\left(\bar{W}+\bar{R}x\right) + tu'\left(\underline{W}+\bar{R}x\right)} - \frac{u'\left(\underline{W}+\underline{R}x\right) - u'\left(\bar{W}+\underline{R}x\right)}{(1-t)\,u'\left(\bar{W}+\underline{R}x\right) + tu'\left(\underline{W}+\underline{R}x\right)} \\
&= \frac{\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)} - 1}{(1-t) + t\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)}} - \frac{\frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)} - 1}{(1-t) + t\frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)}}.
\end{aligned} \tag{27}$$

So (4) holds, since the expression $\frac{y-1}{1-t+ty}$ is increasing in y, and $\frac{u'\left(\underline{W}+\bar{R}x\right)}{u'\left(\bar{W}+\bar{R}x\right)} < \frac{u'\left(\underline{W}+\underline{R}x\right)}{u'\left(\bar{W}+\underline{R}x\right)}$ by DARA, since DARA implies that $\frac{u'(\underline{W}+y)}{u'\left(\bar{W}+y\right)}$ is decreasing in y. These same observations also imply that the first term in (27) is decreasing in x while the second term is increasing in x, so that (5) holds.

B.3 Detailed calculations used in subsection 2.2

By straightforward substitution,

$$\begin{aligned}
\frac{\partial}{\partial \theta}\ln\left(\frac{q_x}{q_\kappa}\right)\Bigg|_{x=x(\theta,t,\kappa)} &= \frac{\partial}{\partial \theta}\ln\left(\frac{U_{qx}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right)}{U_{q\kappa}\left(q\left(x,\theta,\kappa\right),x,\theta,\kappa\right)}\right)\Bigg|_{x=x(\theta,t,\kappa)} \\
&= \frac{U_{qx\theta} + q_\theta U_{qqx}}{U_{qx}} - \frac{U_{q\theta\kappa} + q_\theta U_{qq\kappa}}{U_{q\kappa}} \\
&= \frac{U_{qx\theta} - \frac{U_{q\theta}}{U_{qq}}U_{qqx}}{U_{qx}} - \frac{U_{q\theta\kappa} - \frac{U_{q\theta}}{U_{qq}}U_{qq\kappa}}{U_{q\kappa}}.
\end{aligned}$$

Recall that θ and κ enter U only via the function ψ, and moreover, U is linear in ψ. Accordingly, write U_ψ etc to denote the derivative of U with respect to ψ. Hence $U_{q\theta} = \psi_\theta U_{q\psi}$, $U_{qx\theta} = \psi_\theta U_{qx\psi}$, $U_{q\kappa} = \psi_\kappa U_{q\psi}$, $U_{qq\kappa} = \psi_\kappa U_{qq\psi}$, and $U_{q\theta\kappa} = \psi_\theta \psi_\kappa U_{q\psi\psi} + \psi_{\theta\kappa} U_{q\psi} = \psi_{\theta\kappa} U_{q\psi}$. Hence

$$\begin{aligned}\frac{\partial}{\partial\theta}\ln\left(\frac{q_x}{q_\kappa}\right)\bigg|_{x=x(\theta,t,\kappa)} &= \frac{\psi_\theta U_{qx\psi} - \psi_\theta \frac{U_{q\psi}}{U_{qq}} U_{qqx}}{U_{qx}} - \frac{\psi_{\theta\kappa} U_{q\psi} - \psi_\theta\psi_\kappa \frac{U_{q\psi}}{U_{qq}} U_{qq\psi}}{\psi_\kappa U_{q\psi}} \\ &= -\psi_\theta\left(\frac{U_{q\psi}U_{qqx} - U_{qq}U_{qx\psi}}{U_{qq}U_{qx}} + \frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} - \frac{U_{qq\psi}}{U_{qq}}\right).\end{aligned}$$

I first establish that

$$\frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} - \frac{U_{qq\psi}}{U_{qq}} \geq \frac{\psi_{\theta\kappa}}{\psi_\theta\psi_\kappa} - \frac{1}{\psi}. \tag{28}$$

To ease notation, define $\bar{u}' \equiv u'\left(q\left(\bar{R}-x\right)\right)$ and $\underline{u}' = u'\left(q\left(\underline{R}-x\right)\right)$, with analogous definitions for higher order derivatives. Straightforward differentiation yields

$$\begin{aligned}U_q &= (1-\psi)\left(\bar{R}-x\right)\bar{u}' + \psi\left(\underline{R}-x\right)\underline{u}', \\ U_{qx} &= -(1-\psi)\bar{u}' - \psi\underline{u}' - (1-\psi)\left(\bar{R}-x\right)q\bar{u}'' - \psi\left(\underline{R}-x\right)q\underline{u}'', \\ U_{qq} &= (1-\psi)\left(\bar{R}-x\right)^2\bar{u}'' + \psi\left(\underline{R}-x\right)^2\underline{u}'', \\ U_{q\psi} &= -\left(\left(\bar{R}-x\right)\bar{u}' - \left(\underline{R}-x\right)\underline{u}'\right), \\ U_{qq\psi} &= -\left(\left(\bar{R}-x\right)^2\bar{u}'' - \psi\left(\underline{R}-x\right)^2\underline{u}''\right).\end{aligned}$$

Hence

$$\begin{aligned}-\frac{U_{qq\psi}}{U_{qq}} &= \frac{\left(\bar{R}-x\right)^2\bar{u}'' - \left(\underline{R}-x\right)^2\underline{u}''}{\left(\bar{R}-x\right)^2\bar{u}'' - \psi\left(\left(\bar{R}-x\right)^2\bar{u}'' - \left(\underline{R}-x\right)^2\underline{u}''\right)} \\ &= \frac{\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - 1}{\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - \psi\left(\frac{\left(\bar{R}-x\right)^2\bar{u}''}{\left(\underline{R}-x\right)^2\underline{u}''} - 1\right)}.\end{aligned}$$

Note the function $\frac{y-1}{y-\psi(y-1)}$ is increasing in y, since $y - \psi(y-1) - (y-1)(1-\psi) = 1 > 0$. Hence the function $\frac{y-1}{y-\psi(y-1)}$ varies from $-\frac{1}{\psi}$ to $\frac{1}{1-\psi}$ as y varies from 0 to ∞. Consequently,

$$-\frac{U_{qq\psi}}{U_{qq}} \geq -\frac{1}{\psi},$$

establishing (28).

Second, I evaluate the term

$$\frac{U_{q\psi}U_{qqx}-U_{qq}U_{qx\psi}}{U_{qq}U_{qx}}. \tag{29}$$

The numerator in (29) equals

$$\begin{aligned}
&\left(2\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}''+2\psi\left(\underline{R}-x\right)\underline{u}''+\left(1-\psi\right)\left(\bar{R}-x\right)^{2}q\bar{u}'''+\psi\left(\underline{R}-x\right)^{2}q\underline{u}'''\right)\\
\times\ &\left(\left(\bar{R}-x\right)\bar{u}'-\left(\underline{R}-x\right)\underline{u}'\right)\\
-\ &\left(\left(1-\psi\right)\left(\bar{R}-x\right)^{2}\bar{u}''+\psi\left(\underline{R}-x\right)^{2}\underline{u}''\right)\left(\bar{u}'-\underline{u}'+\left(\bar{R}-x\right)q\bar{u}''-\left(\underline{R}-x\right)q\underline{u}''\right).
\end{aligned} \tag{30}$$

Define

$$\begin{aligned}
A\ =\ &\left(2\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}''+2\psi\left(\underline{R}-x\right)\underline{u}''\right)\left(\left(\bar{R}-x\right)\bar{u}'-\left(\underline{R}-x\right)\underline{u}'\right)\\
-\ &\left(\left(1-\psi\right)\left(\bar{R}-x\right)^{2}\bar{u}''+\psi\left(\underline{R}-x\right)^{2}\underline{u}''\right)\left(\bar{u}'-\underline{u}'\right)\\
=\ &\left(1-\psi\right)\left(\bar{R}-x\right)^{2}\bar{u}''\bar{u}'-\psi\left(\underline{R}-x\right)^{2}\underline{u}''\underline{u}'\\
-\ &2\left(1-\psi\right)\left(\bar{R}-x\right)\left(\underline{R}-x\right)\bar{u}''\underline{u}'+2\psi\left(\underline{R}-x\right)\left(\bar{R}-x\right)\underline{u}''\bar{u}'\\
+\ &\left(1-\psi\right)\left(\bar{R}-x\right)^{2}\bar{u}''\underline{u}'-\psi\left(\underline{R}-x\right)^{2}\underline{u}''\bar{u}',
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
B\ =\ &\left(\left(1-\psi\right)\left(\bar{R}-x\right)^{2}\bar{u}'''+\psi\left(\underline{R}-x\right)^{2}\underline{u}'''\right)\left(\left(\bar{R}-x\right)\bar{u}'-\left(\underline{R}-x\right)\underline{u}'\right)\\
-\ &U_{qq}\left[\left(\bar{R}-x\right)\bar{u}''-\left(\underline{R}-x\right)\underline{u}''\right].
\end{aligned}$$

Then (29) equals

$$\frac{A+Bq}{U_{qx}U_{qq}}.$$

Below, I repeatedly substitute in the individual optimization condition

$$\psi\left(\underline{R}-x\right)\underline{u}'=-\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}'.$$

First,

$$U_{qx}\ =\ -\left(1-\psi\right)\bar{u}'+\left(1-\psi\right)\frac{\bar{R}-x}{\underline{R}-x}\bar{u}'-\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}'q\frac{\bar{u}''}{\bar{u}'}+\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}'q\frac{\underline{u}''}{\underline{u}'}$$

$$
\begin{aligned}
&= -\left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}'\left[\frac{1}{\bar{R}-x}-\frac{1}{\underline{R}-x}+q\frac{\bar{u}''}{\bar{u}'}-q\frac{\underline{u}''}{\underline{u}'}\right]\\
U_{qq} &= \left(1-\psi\right)\left(\bar{R}-x\right)\bar{u}'\left[\left(\bar{R}-x\right)\frac{\bar{u}''}{\bar{u}'}-\left(\underline{R}-x\right)\frac{\underline{u}''}{\underline{u}'}\right].
\end{aligned}
$$

Second

$$
\begin{aligned}
\frac{A}{\left(1-\psi\right)\left(\bar{R}-x\right)} &= \left(\bar{R}-x\right)\bar{u}''\bar{u}'+\left(\underline{R}-x\right)\underline{u}''\underline{u}'\frac{\bar{u}'}{\underline{u}'}\\
&- 2\left(\underline{R}-x\right)\bar{u}''\underline{u}'-2\left(\bar{R}-x\right)\underline{u}''\bar{u}'\frac{\bar{u}'}{\underline{u}'}\\
&+ \left(\bar{R}-x\right)\bar{u}''\underline{u}'+\left(\underline{R}-x\right)\underline{u}''\bar{u}'\frac{\bar{u}'}{\underline{u}'}\\
&= \left(\bar{u}'\right)^2\left(\bar{R}-x\right)\left(-\left(-\frac{\bar{u}''}{\bar{u}'}\right)+2\left(-\frac{\underline{u}''}{\underline{u}'}\right)-\frac{\underline{u}'}{\bar{u}'}\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)\\
&+ \bar{u}'\underline{u}'\left(\underline{R}-x\right)\left(-\left(-\frac{\underline{u}''}{\underline{u}'}\right)+2\left(-\frac{\bar{u}''}{\bar{u}'}\right)-\frac{\bar{u}'}{\underline{u}'}\left(-\frac{\underline{u}''}{\underline{u}'}\right)\right)\\
&= \left(\bar{u}'\right)^2\left(\bar{R}-x\right)\left(2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\left(\frac{\underline{u}'}{\bar{u}'}-1\right)\right)\\
&+ \bar{u}'\underline{u}'\left(\underline{R}-x\right)\left(-2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)+\left(-\frac{\underline{u}''}{\underline{u}'}\right)\left(1-\frac{\bar{u}'}{\underline{u}'}\right)\right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{A}{\left(1-\psi\right)\left(\bar{u}'\right)^2\left(\bar{R}-x\right)^2} &= 2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\left(\frac{\underline{u}'}{\bar{u}'}-1\right)\\
&- \frac{1-\psi}{\psi}\left(-2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)+\left(-\frac{\underline{u}''}{\underline{u}'}\right)\left(1-\frac{\bar{u}'}{\underline{u}'}\right)\right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{A}{\frac{1-\psi}{\psi}\left(\bar{u}'\right)^2\left(\bar{R}-x\right)^2} &= 2\psi\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\psi\left(-\frac{\bar{u}''}{\bar{u}'}\right)\left(\frac{\underline{u}'}{\bar{u}'}-1\right)\\
&+ \left(1-\psi\right)\left(2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\left(-\frac{\underline{u}''}{\underline{u}'}\right)\left(1-\frac{\bar{u}'}{\underline{u}'}\right)\right),\\
&= 2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\psi\left(-\frac{\bar{u}''}{\bar{u}'}\right)\left(\frac{\underline{u}'}{\bar{u}'}-1\right)-\left(1-\psi\right)\left(-\frac{\underline{u}''}{\underline{u}'}\right)\left(1-\frac{\bar{u}'}{\underline{u}'}\right).
\end{aligned}
$$

Combining,

$$\frac{A}{U_{qx}U_{qq}} = -\frac{1}{\psi}\frac{2\left(-\frac{\underline{u}''}{\underline{u}'}-\left(-\frac{\bar{u}''}{\bar{u}'}\right)\right)-\psi\left(-\frac{\bar{u}''}{\bar{u}'}\right)\left(\frac{\underline{u}'}{\bar{u}'}-1\right)-(1-\psi)\left(-\frac{\underline{u}''}{\underline{u}'}\right)\left(1-\frac{\bar{u}'}{\underline{u}'}\right)}{\left(\frac{1}{\bar{R}-x}-\frac{1}{\underline{R}-x}+q\frac{\bar{u}''}{\bar{u}'}-q\frac{\underline{u}''}{\underline{u}'}\right)\left(\left(\bar{R}-x\right)\frac{\bar{u}''}{\bar{u}'}-\left(\underline{R}-x\right)\frac{\underline{u}''}{\underline{u}'}\right)},$$

which converges to 0 as $s(0), s(1) \to 0$, since in this case q approaches 0 for all realizations of t, so that $\frac{\bar{u}'}{\underline{u}'} \to 1$ and $\left|\frac{\underline{u}''}{\underline{u}'} - \frac{\bar{u}''}{\bar{u}'}\right| \to 0$, and $x \to (1-\psi)\bar{R} + \psi\underline{R}$.

Moreover, the same observations imply that, for any realization of t,

$$\frac{Bq}{U_{qx}U_{qq}} \to 0 \text{ as } s(0), s(1) \to 0.$$

Hence expression (29) approaches 0 as $s(0), s(1) \to 0$.

B.4 Comparison of log-submodularity of ψ and log-supermodularity of the likelihood ratio $\frac{1-\psi}{\psi}$

Note that log-submodularity of ψ implies log-supermodularity of the likelihood ratio $\frac{1-\psi}{\psi}$, as follows. Log supermodularity of $\frac{1-\psi}{\psi}$ is equivalent to

$$\frac{\partial}{\partial\theta}\left(\frac{-\psi_\kappa}{1-\psi} - \frac{\psi_\kappa}{\psi}\right) \geq 0,$$

i.e.,

$$\frac{\psi_{\kappa\theta}(1-\psi)+\psi_\kappa\psi_\theta}{(1-\psi)^2} + \frac{\psi_{\kappa\theta}\psi - \psi_\kappa\psi_\theta}{\psi^2} \leq 0,$$

i.e.,

$$(1-\psi)\psi\psi_{\kappa\theta} + \left(\psi^2 - (1-\psi)^2\right)\psi_\kappa\psi_\theta \leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta} + \frac{2\psi-1}{1-\psi}\psi_\kappa\psi_\theta \leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta} - \psi_\kappa\psi_\theta + \left(\frac{2\psi-1}{1-\psi}+1\right)\psi_\kappa\psi_\theta \leq 0,$$

i.e.,

$$\psi\psi_{\kappa\theta} - \psi_\kappa\psi_\theta + \frac{\psi}{1-\psi}\psi_\kappa\psi_\theta \leq 0.$$

B.5 Detailed calculations used in subsection 2.3

B.5.1 Verification of the equilibrium price (15)

Case: $\theta < -\frac{s(t)}{\lambda} - K$

The claimed equilibrium price is $x = \frac{\theta - s(t) + K}{\lambda+1}$.

Note that $\lambda(\theta + K) + s(t) < 0$.

So informed agent demand is $\theta + K - x = \theta + K - \frac{\theta - s(t)+K}{\lambda+1} = \frac{\lambda(\theta+K)+s(t)}{\lambda+1} < 0$, and noise demand is $-(s(t) + \lambda x) = -\left(s(t) + \lambda\frac{\theta - s(t)+K}{\lambda+1}\right) = -\left(\frac{s(t)+\lambda(\theta+K)}{\lambda+1}\right)$, so that the market indeed clears.

Case: $\theta \in \left[-\frac{s(t)}{\lambda} - K, -\frac{s(t)}{\lambda}\right]$

The claimed equilibrium price is $x = -\frac{s(t)}{\lambda}$.

So $\theta \in [x - K, x]$, and informed agent demand is 0.

Noise trader demand is likewise 0, so that the market indeed clears.

Case: $\theta > -\frac{s(t)}{\lambda}$

The claimed equilibrium price is $x = \frac{\theta - s(t)}{\lambda+1}$.

Note that $\lambda\theta + s(t) > 0$.

So informed agent demand is $\theta - x = \theta - \frac{\theta - s(t)}{\lambda+1} = \frac{\lambda\theta + s(t)}{\lambda+1} > 0$, and noise demand is $-(s(t) + \lambda x) = -\left(s(t) + \lambda\frac{\theta - s(t)}{\lambda+1}\right) = -\left(\frac{s(t)+\lambda\theta}{\lambda+1}\right)$, so that the market indeed clears.

B.5.2 SCP

First note that the equilibrium price (15) is continuous: if $\theta = -\frac{s}{\lambda}$ then $\frac{\theta - s}{\lambda+1} = -\frac{s}{\lambda}$, and if $\theta = -\frac{s}{\lambda} - K$ then $\frac{\theta - s + K}{\lambda+1} = -\frac{s}{\lambda}$. So the equilibrium price can be concisely expressed as

$$x(\theta, t) = \min\left\{\frac{\theta - s(t) + K}{\lambda+1}, \max\left\{-\frac{s(t)}{\lambda}, \frac{\theta - s(t)}{\lambda+1}\right\}\right\}.$$

I next establish that x satisfies SCP in $((\theta, t); \kappa)$. That is: If $\theta_2 \geq \theta_1$, $t_2 \geq t_1$, $\kappa_2 > \kappa_1$, $x(\theta_2, t_2; \kappa_1) \geq (>) x(\theta_1, t_1; \kappa_1)$ then $x(\theta_2, t_2; \kappa_2) \geq (>) x(\theta_1, t_1; \kappa_2)$.

The proof is immediate if $t_2 = t_1$. Consider next the case of $t_2 > t_1$. There are three cases:

Case (i): $x(\theta_2, t_2; \kappa_1) = \frac{\theta_2 - s(t_2) + K(\kappa_1)}{\lambda+1}$ and $x(\theta_1, t_1; \kappa_1) = \frac{\theta_1 - s(t_1) + K(\kappa_1)}{\lambda+1}$.

In this case, $x(\theta_2, t_2; \kappa_2) = \frac{\theta_2 - s(t_2) + K(\kappa_2)}{\lambda+1}$ and $x(\theta_1, t_1; \kappa_2) = \frac{\theta_1 - s(t_1) + K(\kappa_2)}{\lambda+1}$, and the result follows.

Case (ii): $x(\theta_2, t_2; \kappa_1) = \max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda+1}\right\}$ and $x(\theta_1, t_1; \kappa_1) = \frac{\theta_1 - s(t_1) + K(\kappa_1)}{\lambda+1}$.

In this case,

$$\frac{\theta_2 - s(t_2) + K(\kappa_1)}{\lambda + 1} \geq \max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda + 1}\right\} \geq (>) \frac{\theta_1 - s(t_1) + K(\kappa_1)}{\lambda + 1}.$$

So $x(\theta_2, t_2; \kappa_2) = \min\left\{\frac{\theta_2 - s(t_2) + K(\kappa_2)}{\lambda+1}, \max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda+1}\right\}\right\}$ and $x(\theta_1, t_1; \kappa_2) = \frac{\theta_1 - s(t_1) + K(\kappa_2)}{\lambda+1}$, and the result follows.

Case (iii): $x(\theta_1, t_1; \kappa_1) = \max\left\{-\frac{s(t_1)}{\lambda}, \frac{\theta_1 - s(t_1)}{\lambda+1}\right\}$.

Since $t_2 > t_1$, $-\frac{s(t_2)}{\lambda} < -\frac{s(t_1)}{\lambda}$. In this case $x(\theta_2, t_2; \kappa_1) \geq (>) x(\theta_1, t_1; \kappa_1)$ implies $\max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda+1}\right\} \geq x(\theta_2, t_2; \kappa_1) \geq (>) \max\left\{-\frac{s(t_1)}{\lambda}, \frac{\theta_1 - s(t_1)}{\lambda+1}\right\}$. Consequently, $\frac{\theta_2 - s(t_2)}{\lambda+1} \geq (>) \frac{\theta_1 - s(t_1)}{\lambda+1}$. Hence

$$\begin{aligned} \frac{\theta_2 - s(t_2) + K(\kappa_2)}{\lambda + 1} &\geq (>) \quad \frac{\theta_1 - s(t_1) + K(\kappa_2)}{\lambda + 1} \geq x(\theta_1, t_1; \kappa_2) \\ \max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda + 1}\right\} &\geq (>) \quad \max\left\{-\frac{s(t_1)}{\lambda}, \frac{\theta_1 - s(t_1)}{\lambda + 1}\right\} \geq x(\theta_1, t_1; \kappa_2), \end{aligned}$$

which implies

$$x(\theta_2, t_2; \kappa_2) = \min\left\{\frac{\theta_2 - s(t_2) + K(\kappa_2)}{\lambda + 1}, \max\left\{-\frac{s(t_2)}{\lambda}, \frac{\theta_2 - s(t_2)}{\lambda + 1}\right\}\right\} \geq (>) \, x(\theta_1, t_1; \kappa_2),$$

completing the proof.