SINGULAR PERTURBATION METHODS FOR NONLINEAR SHOCK ABSORBER: METHOD OF MULTIPLE SCALES
1. **INTRODUCTION**

This report contains a summary on utilizing singular perturbation methods to approximate the behavior of a simplified automobile shock absorber. In the paragraphs that follow, the physical system shall be discussed, a dynamic model defined, and the singular perturbation methods used to solve approximate the model shall be presented. Furthermore, the results of the given analysis will be presented and discussed in detail.
2. **SHOCK ABSORBER DESCRIPTION – DEFINITIONS AND PRINCIPALS**

   Typical automobile shock absorbers combine hydraulic damping with spring deflection to absorb the kinetic energy associated with the mass of a moving car. The “shock” consists of a cylinder and a piston which slides freely into and out of the cylinder. While there are several variations of this design, the key feature is that hydraulic fluid flows through small passage ways in the cylinder as the shock compresses or extends. For the current model (see Figure 2), the top of the piston, or “piston head”, separates the upper and lower chamber within the cylinder. The piston head also houses small holes called “orifices” which allow fluid flow between the two chambers.

   In order for flow to exist, a pressure differential must exist across the piston head. This pressure difference is caused by relative motion between the mass connected to the top of the shock and the lower joint of the shock. As the car drives over a bumpy surface, force is exerted on the tire and thus the lower chamber is pressurized.
3. **ASYMPTOTIC EXPANSION**

Given the nonlinear damping associated with hydraulic shock absorbers, asymptotic expansion methods were proven useful in determining approximate solutions to the dynamics. All methods used are discussed herein and the asymptotic solutions results compared to standard numerical solutions (RK-4 scheme).

### 3.1 Shock Absorber Dynamics

In order to simplify the dynamics, only the axial motion of the shock absorber was considered. Furthermore, the shock was given only one degree of freedom corresponding to the motion of the supported mass. The dynamics, derived in Appendix A, are given by:

\[
m\ddot{y} + \frac{\rho}{2} \frac{A_c^3}{(C_d A_o)^2} \dot{y}|y| + ky = 0
\]  

[3-1]

Which can be simplified to:

\[
\ddot{y} + \varepsilon \dot{y}|\dot{y}| + \omega^2 y = 0
\]  

[3-2]

Where \( \varepsilon = \frac{\rho}{2m (C_d A_o)^2} \) and \( \omega^2 = \frac{k}{m} \).
3.2 Breakdown of Standard Expansion

If a standard power series expansion is assumed, the solution may be written as:

\[ y(t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n(t) \]  

Or up to \( O(\varepsilon) \):

\[ y \sim y_0 + \varepsilon y_1 \]  

Here, we say that \( y \) is asymptotically equal to \( y_0 + \varepsilon y_1 \), the first two terms in the power series expansion. Plugging this solution into the differential equation yields:

\[ (\ddot{y}_0 + \varepsilon \ddot{y}_1) + \varepsilon (\dot{y}_0 + \varepsilon \dot{y}_1) \left| \dot{y}_0 + \varepsilon \dot{y}_1 \right| + \omega^2 (y_0 + \varepsilon y_1) = 0 \]

This gives the \( O(1) \) equation:

\[ \ddot{y}_0 + \omega^2 y_0 = 0 \]  

Whose general solution is:

\[ y_0 = a \cos(\omega t + \beta) \]  

The \( O(\varepsilon) \) equation is then:

\[ \dot{y}_1 + \omega^2 y_1 = -\dot{y}_0 \left| \dot{y}_0 \right| = f(\dot{y}_0) \]  

To obtain analytic solutions, the Fourier Series representation of the periodic function \( -\dot{y}_0 \left| \dot{y}_0 \right| = a^2 \omega^2 \sin(\omega t + \beta) \left| \sin(\omega t + \beta) \right| \) may be used. Given that this function has a period of \( \frac{2\pi}{\omega} \), its Fourier Series is given by:

\[ f(\dot{y}_0) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t + n\beta) + b_n \sin(n\omega t + n\beta)) \]  

Where the coefficients are defined as:

\[ a_0 = \frac{\omega}{2\pi} \int_{0}^{\frac{2\pi}{\omega}} f(\dot{y}_0) \, dt \]

\[ a_n = \frac{\omega}{\pi} \int_{0}^{\frac{2\pi}{\omega}} f(\dot{y}_0) \cos(n\omega t + n\beta) \, dt \]  

\[ b_n = \frac{\omega}{\pi} \int_{0}^{\frac{2\pi}{\omega}} f(\dot{y}_0) \sin(n\omega t + n\beta) \, dt \]

Upon calculating these coefficients and rewriting the \( O(\varepsilon) \) equation, it becomes apparent that the particular solution for \( y_1 \) yields a non-uniform expansion. That is, assuming that the coefficients \( a_1 \) and \( b_1 \)
corresponding to \( \cos(\omega t + \beta) \) and \( \sin(\omega t + \beta) \) respectively, are nonzero, the particular solution for \( y_1 \) (Nayfeh, 1993) will have the form:

\[
y_{1,p} = a_0 + \frac{1}{2} a_1 t \sin(\omega t + \beta) - \frac{1}{2} b_1 t \cos(\omega t + \beta) + \sum_{n=2}^{\infty} \frac{1}{1 - n^2} [a_n \cos(n\omega t + n\beta) + b_n \sin(n\omega t + n\beta)]
\]  

[3-11]

Given [3-4], we see that this solution breaks down at \( t = O(\varepsilon^{-1}) \).

### 3.3 Method of Multiple Scales

To avoid the singularity described in 3.2, the method of multiple scales was utilized. In using this method, two time scales were defined as \( T_0 = t \) and \( T_1 = \varepsilon t \). These scales were assumed independent, rendering a new expansion:

\[
y(T_0, T_1, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y_n(T_0, T_1)
\]  

[3-12]

With time derivatives given by:

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} = D_0 + \varepsilon D_1
\]  

[3-13]

\[
\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2}{\partial T_1^2} + D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_1^2
\]  

[3-14]

The equation of motion, [3-2], becomes:

\[
(D_0^2y + 2\varepsilon D_0 D_1 y + \varepsilon^2 D_0^2 y) + \varepsilon(D_0 y + \varepsilon D_1 y) |D_0 y + \varepsilon D_1 y| + \omega^2 y = 0
\]  

[3-15]

Or, expanding to \( O(\varepsilon) \):

\[
(D_0^2y_0 + \varepsilon D_0 y_1 + 2\varepsilon D_0 D_1 y_0 + \ldots) + \varepsilon(D_0 y_0 + \ldots) |D_0 y_0 + \ldots| + \omega^2(y_0 + \varepsilon y_1) = 0
\]  

[3-16]

This gives the \( O(1) \) equation:

\[
D_0^2y_0 + \omega^2 y_0 = 0
\]  

[3-17]

Whose general solution is:

\[
y_0 = a \cos(\omega T_0 + \beta)
\]  

[3-18]

Where \( a = a(T_1) \) and \( \beta = \beta(T_1) \). The partial derivatives are:

\[
D_0 y_0 = -a\omega \sin(\omega T_0 + \beta)
\]  

[3-19]

\[
D_1 D_0 y_0 = -a'\omega \sin(\omega T_0 + \beta) - a\omega \beta' \cos(\omega T_0 + \beta)
\]  

[3-20]

Given the initial conditions \( y_0(0) = 0 \) and \( D_0 y_0(0) = \dot{y}_0 \), it follows that \( a_0 \cos \beta_0 = 0 \) and \( a_0 \sin \beta_0 = \frac{\dot{y}_0}{\omega} \). Or:
\[
\beta_0 = \frac{\pi}{2}
\]
\[
a_0 = -\frac{\dot{y}_0}{\omega}
\]

The \(O(\varepsilon)\) equation is then:

\[
D_0^2 \ddot{y}_1 + \omega^2 y_1 = -2D_0D_1y_0 - D_0y_0 |D_0y_0|
\]

\[
D_0^2 \ddot{y}_1 + \omega^2 y_1 = 2a'\omega \sin(\omega T_0 + \beta) + 2a\omega\beta' \cos(\omega T_0 + \beta) + a^2 \omega^2 \sin(\omega T_0 + \beta) |\sin(\omega T_0 + \beta)|
\]  

Similar to the straightforward expansion, the Fourier Series representation of \(f(D_0y_0) = a^2 \omega^2 \sin(\omega T_0 + \beta) |\sin(\omega T_0 + \beta)|\) may be used to obtain analytic solutions. By making the substitution \(\phi = \omega T_0 + \beta\), the Fourier Series with respect to \(\phi\) is given by:

\[
f(D_0y_0) = f_0(a) + \sum_{n=1}^{\infty} (f_n(a) \cos(n\phi) + g_n(a) \sin(n\phi))
\]

Where the coefficient functions are given by:

\[
f_0(a) = \frac{1}{2\pi} \int_{0}^{2\pi} f(D_0y_0) \, d\phi
\]

\[
f_n(a) = \frac{1}{\pi} \int_{0}^{2\pi} f(D_0y_0) \cos(n\phi) \, d\phi
\]

\[
g_n(a) = \frac{1}{\pi} \int_{0}^{2\pi} f(D_0y_0) \sin(n\phi) \, d\phi
\]

Substituting \(f(D_0y_0) = a^2 \omega^2 \sin\phi |\sin\phi|\) and noting that \(\sin\phi > 0\) for \(0 \leq \phi \leq \pi\) and \(\sin\phi < 0\) for \(\pi \leq \phi \leq 2\pi\) gives:

\[
f_0(a) = \frac{a^2 \omega^2}{2\pi} \left( \int_{0}^{\pi} \sin^2 \phi \, d\phi - \int_{\pi}^{2\pi} \sin^2 \phi \, d\phi \right)
\]

\[
f_n(a) = \frac{a^2 \omega^2}{\pi} \left( \int_{0}^{\pi} \sin^2 \phi \cos(n\phi) \, d\phi - \int_{\pi}^{2\pi} \sin^2 \phi \cos(n\phi) \, d\phi \right)
\]

\[
g_n(a) = \frac{a^2 \omega^2}{\pi} \left( \int_{0}^{\pi} \sin^2 \phi \sin(n\phi) \, d\phi - \int_{\pi}^{2\pi} \sin^2 \phi \sin(n\phi) \, d\phi \right)
\]

In order to remove the secular terms from the right hand side of [3-22], it is required that:

\[2a' \omega + g_1(a) = 0 \quad \text{and} \quad 2a\omega\beta' + f_1(a) = 0\]
Upon evaluating the integrals $g_1(a)$ and $f_1(a)$, the following equalities are obtained:

$$2a'\omega + \frac{8a^2\omega^2}{3\pi} = 0 \quad \text{and} \quad 2a\omega\beta = 0$$

This results in $\beta' = 0$, or $\beta = \text{constant} = \beta_0 = \frac{\pi}{2}$ (see [3-21]). Integrating the first equation yields:

$$a(T_1) = \frac{3\pi a_0}{3\pi + 4\omega a_0 T_1} = \frac{-3\pi \dot{y}_0}{3\pi - 4\omega \dot{y}_0 T_1} \quad [3-26]$$

Or in terms of $T_0$:

$$a(T_0) = \frac{-3\pi \dot{y}_0}{3\pi - 4\omega \dot{y}_0 \varepsilon T_0} \quad [3-27]$$

Substituting $a(T_0) = a(t)$ and $\beta(T_0) = \frac{\pi}{2}$ in Error! Reference source not found. gives the first order asymptotic expansion:

$$y(t) \sim \frac{-3\pi \dot{y}_0}{3\pi - 4\omega \dot{y}_0 \varepsilon t} \cos(\omega t + \frac{\pi}{2}) \quad [3-28]$$

4. RESULTS

Upon finding the $O(1)$ expansion, the dynamics were solved numerically to compare the two solutions. In doing so, the initial conditions, $\dot{y}_0 = D_0 y_0 (0)$, were varied from -1 to -100 to determine sensitivity. As seen in Figure 3 and Figure 4, the $O(1)$ expansion matched the numerical solution extremely well for low initial velocities. However, Figure 5 demonstrates higher initial errors at high initial velocity ($\dot{y}_0 = -100$). This was intuitive and undoubtedly due to the perturbation dependency on $\dot{y}$ (see [3-2]).

In addition to varying $\dot{y}_0$, $\varepsilon$ was varied between 0.05 to 0.5 to analyze robustness of the expansion. As expected, the error tended to increase with increasing $\varepsilon$. It is worth noting that the max error decreased between $\varepsilon = 0.1$ and $\varepsilon = 0.5$ for the case of $\dot{y}_0 = -100$. However, relative to the numerical solution values, the max error for these cases corresponded to 15.2% and 32.3% error for $\varepsilon = 0.1$ and $\varepsilon = 0.5$ respectively.
Figure 3: $O(1)$ expansion vs RK-4 solution, $\dot{y}(0) = -1$

Figure 4: $O(1)$ expansion vs RK-4 solution, $\dot{y}(0) = -10$
Figure 5: $O(1)$ expansion vs RK-4 solution, $\dot{y}(0) = -100$

Figure 6: Error of $O(1)$ expansion vs RK-4 solution, $\varepsilon = 0.05$
Figure 7: Error of $O(1)$ expansion vs RK-4 solution, $\varepsilon = 0.1$

Figure 8: Error of $O(1)$ expansion vs RK-4 solution, $\varepsilon = 0.5$
5. CONCLUSIONS
Upon analyzing the results, it was found that for all cases, the first order expansions approached the numerical solutions as time increased \((t \to \infty)\). This is demonstrated in the plots of error versus time. Furthermore, for low initial velocities the O(1) expansion matched numerical solutions extremely well, even for cases of increasing \(\varepsilon\). Given the success of the O(1) expansion, the addition of higher order terms was not deemed necessary; however, the same procedure could be implemented to determine \(O(\varepsilon)\) or higher terms.

6. BIBLIOGRAPHY

APPENDIX A: DERIVING EQUATION OF MOTION
Lagrange’s Equation of Motion can be derived either from the 1st Law of Thermodynamics in conjunction with analytical dynamic principles, or Hamilton’s Principle of Least Action (Fabien, 2009). It can be generalized to a multidiscipline system as follows:

\[
\frac{d}{dt} \left( \frac{\partial T^*}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} - e_{q_i} = 0, \quad i = 1,2,3, \ldots n
\]  

[A-1]

In this equation \(q_i\) is the \(i^{th}\) generalized (independent) displacement variable of the dynamic system, \(T^*\) is the kinetic energy of the system, \(V\) is the potential energy of the system, and \(D\) is the energy dissipation function. In order to solve this equation in the form seen above, each of its terms must be expressed as functions of the \(n\) generalized displacement variables. The term \(e_{q_i}\) is the “effort” applied to each displacement variable. For example, in mechanical translation, the effort is equal to the force applied to the \(i^{th}\) displacement variable of the system.

For the current shock absorber system, the schematic below was used to define the generalized coordinates. The system has one degree of freedom as seen in Figure 9.
A-1 Kinetic Energy
The kinetic energy of an inertial element is a function of the “generalized flow” associated with that element. From Figure 9, it is seen that the total kinetic energy of the system is:

\[ T = \frac{1}{2} m \dot{y}^2 \]  

\[ \text{[A-2]} \]

A-2 Potential Energy
The total potential energy of the system includes the potential due to gravity as well as the energy stored in the two linear springs. It is given by:

\[ V = \frac{1}{2} k (y + y_0)^2 - mgy \]  

\[ \text{[A-3]} \]

A-3 Applied Effort
The work performed on the fluid can be expressed in terms of the pressure drop over the piston head. A well know equation for orifice flow gives the pressure drop over any orifice as a function of fluid flow. It is derived from Bernoulli’s Equation and is given by:

\[ \Delta P = \frac{\rho}{2} \left( \frac{Q}{C_d A_o} \right)^2 \]  

\[ \text{[A-4]} \]

Where \( Q \) is the volumetric flow rate, \( \rho \) is the density of the fluid, \( C_d \) is a dimensionless value called the coefficient of discharge, and \( A_o \) is the orifice area.

The primary energy absorption characteristics of an automobile shock absorber are a result of fluid flow through small orifices between the upper and lower chambers. As the shock compresses or extends, the kinetic energy present in the moving fluid is dissipated as heat. This dissipated energy may be quantified by calculating the work performed on the fluid. In terms of the damping force, the work done on the fluid over the piston head is given by:
Here, \( \text{Stroke} \) is the relative displacement between the two chambers \( F_{\text{hyd}} \) is the net hydraulic force over the piston head. In terms of the pressure drop over the piston head, the work can be expressed as:

\[
W_{\text{hyd}} = \Delta P \times \text{Fluid Area} \times \text{Stroke}
\]  

For this system, the hydraulic damping was represented as an applied force rather than a dissipation function. Applying the principal of virtual work, the total variation in work done on the system at a moment in time is expressed as:

\[
\delta W = -\Delta P \delta V_{\text{fluid}}
\]  

Here, \( \Delta P \) is the pressure drop over the piston head and \( \delta V_{\text{fluid}} \) is the variation in fluid volume. Notice that the work done on the fluid carries a negative sign because it is dissipated as heat.

If the fluid flow is assumed incompressible, the volumetric flow rate is given by \( Q = A_c \dot{s} \) and the volume variation by \( \delta V_{\text{fluid}} = A_c \delta s \), where \( A_c \) is the inner cylinder area and \( \dot{s} \) is the stroke rate. Rewriting \( \dot{s} \) and \( \delta s \) in terms of the generalized coordinates gives:

\[
\dot{s} = \dot{y}
\]

\[
\delta s = \delta y
\]

In terms of the generalized coordinates, the variation in work done on the system is then:

\[
\delta W = -\frac{\rho}{2} \frac{A_c^3}{(c_d A_o)^2} \dot{y}^2 \delta y
\]

Note: Equation Error! Reference source not found. was derived under the assumption that the shock undergoes compression. Because of the term \( \dot{y}^2 \), the sign of equation Error! Reference source not found. must be flipped when \( \dot{y}^2 < 0 \). This is easily accomplished during the numerical solution process and shall be discussed in section Error! Reference source not found..

**A-3 Equation of Motion – Vertical Displacement of Sprung Mass**

In reference to Lagrange’s Equation, each term is tabulated below for the variable \( y_1 \), the vertical displacement of the sprung mass.

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial y_1} \right) = m\ddot{y}
\]

\[
\frac{\partial T}{\partial y} = 0
\]
\[
\frac{\partial V}{\partial y} = k(y + y_0) - mg \tag{A-13}
\]
\[
\frac{\partial D}{\partial \dot{y}} = 0 \tag{A-14}
\]
\[
e_y = -\frac{\rho}{2} \cdot \frac{A_c^3}{(C_d A_o)^2} \dot{y}^2 \tag{A-15}
\]
Combining terms:
\[
m\ddot{y} + \frac{\rho}{2} \cdot \frac{A_c^3}{(C_d A_o)^2} \dot{y}^2 + k(y + y_0) - mg = 0 \tag{A-16}
\]
Or, starting at an equilibrium position:
\[
m\ddot{y} + \frac{\rho}{2} \cdot \frac{A_c^3}{(C_d A_o)^2} \dot{y}^2 + ky = 0 \tag{A-17}
\]

**APPENDIX B: MATLAB CODE**

```matlab
% AA598: Singular Perturbation Theory - Final Project
% "Singular Perturbation Methods for Nonlinear Shock Absorber"
% 6/10/2014
% By: Michael Czerski
% This script applies the method of multiple scales to compute the
% asymptotic expansion of the non-analytic differential equation associated
% with a nonlinear shock absorber. In doing so, the O(1) term is solved and
% plugged into the O(epsilon) equation. The absolute value term is
% represented as a Fourier series and used to find the solutions for a(T1)
% and Beta(T1), defined by the scaling T1 = epsilon*T0.

clear all, clc

% Define parameters
beta0 = pi/2;
dt = 0.05;
t = 0:dt:10;
w = 3; % Undamped natural frequency
eps = 0.1; % Perturbation rho/(2*M)*(Ac^3/((Cd*A0)^2));
eps = 0.05; % Try different values
eps = 0.5;

% Calculate O(1) term in asymptotic expansion of differential equation
y'' + eps*(y')^2 + w^2*y = 0; y(0) = 0, y'(0) = -1.
syms omega T0 phi a T1 dy0 real
y0init = 'y0(0)=0, Dy0(0)=dy0';
```
deqn_y0 = 'D2y0 + omega^2*y0 = 0';

y0sol = simplify(dsolve(deqn_y0,y0init,'T0')); %beta0=pi/2 & a0=-dy0/w

% Calculate coefficients of fourier series representation of
% f = (w*a)^2*sin(phi)*abs(sin(phi));
% Note: fn not integrated since f0 = f1 = .... = fn = 0 for all n!

F = sin(phi)*sin(phi);
% f0lsym = 1/(2*pi)*(int(F,phi, 0 , pi) - int(F,phi, pi, 2*pi));

N = 11;
for n = 1:N
    fsym(n) = 1/(pi)*(int(F*cos(n*phi),phi, 0 , pi) -...
                     int(F*cos(n*phi),phi, pi, 2*pi));
    gsym(n) = (omega*a)^2/(pi)*(int(F*sin(n*phi),phi, 0 , pi) -...
                               int(F*sin(n*phi),phi, pi, 2*pi));
end
g1sec = gsym(1);
% f1sec = fsym(1);
gsym(1) = 0;
% fsym(1) = 0;
fsym = zeros(size(gsym));

% Solve ode da/dT1 = -4*omega*a^2/(3*pi); a(0) = -y'(0)/omega. Where I.C. was
% found analytically by setting secular terms equal to zero in O(epsilon)
% equation.

% Initial condition
inits1 = 'a(0)=-dy0/omega';

% Differential equation
deqn_a = 'Da = -4*omega*a^2/(3*pi)';

% Solve w/ dsolve() in terms of T1 = epsilon*T0.
a_T1 = dsolve(deqn_a,inits1,'T1');

% Define in terms of T0.
a_T0 = subs(a_T1,T1,eps*T0);

% Calculate O(1) term in terms of a(T0) and dy/dt(0)
y0_T0 = a_T0*(cos(w*T0+beta0));
y0_dy0 = subs(y0_T0,omega,w);

% Solve O(1) and O(eps) terms, y0 & y1, for different initial condition
% dy/dt(0).
% dy/dt(0) = -1.
y0_1 = subs(y0_dy0,dy0,-1);
y0_1 = subs(y0_1,T0,t);

% dy/dt(0) = -10.
y0_2 = subs(y0_dy0,dy0,-10);
y0_2 = subs(y0_2,T0,t);

% dy/dt(0) = -100.
y0_3 = subs(y0_dy0,dy0,-100);
y0_3 = subs(y0_3,T0,t);

y1 = y0_1;
y2 = y0_2;
y3 = y0_3;

%% Compare with direct numerical solution using ode45 and ode function
% Singular_Damping_ode.
yinit = [0;-1];
[t,Y1] = ode45(@ (t,y) Singular_Damping_ode(t,y,eps,w,0,0),t,yinit);

figure(1)
pplot(t,Y1(:,1),t,y1,'ro')
title('Shock Absorber Dynamics: dy/dt(0) = -1, \epsilon = 0.1, \omega = 3','FontSize',18)
xlabel('Time [s]','FontSize',14)
ylabel('Position','FontSize',14)
legend('Numerical Solution','1^{st} Order Expansion')
ggrid on

%% Try varying I.C.'s to check robustness of expansion.

% dy/dt(0) = -10.
yinit = [0;-10];
[t,Y2] = ode45(@ (t,y) Singular_Damping_ode(t,y,eps,w,0,0),t,yinit);

figure(2)
pplot(t,Y2(:,1),t,y2,'ro')
title('Shock Absorber Dynamics: dy/dt(0) = -10, \epsilon = 0.1, \omega = 3','FontSize',18)
xlabel('Time [s]','FontSize',14)
ylabel('Position','FontSize',14)
legend('Numerical Solution','1^{st} Order Expansion')
ggrid on

% dy/dt(0) = -100.
yinit = [0;-100];
[t,Y3] = ode45(@ (t,y) Singular_Damping_ode(t,y,eps,w,0,0),t,yinit);

figure(3)
pplot(t,Y3(:,1),t,y3,'ro')
title('Shock Absorber Dynamics: dy/dt(0) = -100, \epsilon = 0.1, \omega = 3','FontSize',18)
xlabel('Time [s]','FontSize',14)
ylabel('Position','FontSize',14)
legend('Numerical Solution','1^{st} Order Expansion')
grid on

%% Plot error between O(1) expansion and numerical solution for each I.C.
err1 = Y1(:,1)-y1';
err2 = Y2(:,1)-y2';
err3 = Y3(:,1)-y3';
[me1,I1] = max(abs(err1));
[me2,I2] = max(abs(err2));
[me3,I3] = max(abs(err3));
me1 = me1/abs(Y1(I1,1))*100;
me2 = me2/abs(Y2(I2,1))*100;
me3 = me3/abs(Y3(I3,1))*100;

L1 = strcat('\frac{dy}{dt}(0) = -1; \text{ Max Error} = ',num2str(me1));
L2 = strcat('\frac{dy}{dt}(0) = -10; \text{ Max Error} = ',num2str(me2));
L3 = strcat('\frac{dy}{dt}(0) = -100; \text{ Max Error} = ',num2str(me3));
figure(5)
plot(t,err1,t,err2,t,err3)
title('Error Between O(1) & Numerical Solution: & \epsilon = 0.1, \omega = 3', 'FontSize',18)
xlabel('Time [s]', 'FontSize',14)
ylabel('Error', 'FontSize',14)
grid on
legend(L1,L2,L3)

function [ dYdt ] = Singular_Damping( t,y,eps,w,Xin,dXin )
% Singular_Damping defines the nonlinear differential equations of motion
% (EOM's) associated with an isolated automobile shock absorber w/ one
% degree of freedom (supported mass). It accepts the parameters
% epsilon(perturbation term) and w(undamped natural frequency). The
% parameters Xin and dXin allow user to apply input, i.e. the motion of
% the shock's lower spherical joint is included as an independent function
% of time. In the equations that follow, y(1) and y(2) and the position and
% velocity of the supported mass respectively.

s = y(1)+Xin; %Shock stroke
sdot = y(2)+dXin; %Stroke rate

if sdot > 0 %Define EOM's during compression
    dy1 = y(2); %dy1 = y2, velocity
    dy2 = -eps*sdot^2 - w^2*s; %dy2 = acceleration. damping term > 0
    dYdt = [dy1 ; dy2];
else %Define EOM's during extension
    dy1 = y(2); %dy1 = y2, velocity
    dy2 = eps*sdot^2 - w^2*s; %dy2 = acceleration. damping term < 0
    dYdt = [dy1 ; dy2];
end